

Dual linearized gravity in $D = 6$ coupled to a purely spin-two field of mixed symmetry $(2, 2)$

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January 13, 2013

Abstract

Under the hypotheses of analyticity, locality, Lorentz covariance, and Poincaré invariance of the deformations, combined with the requirement that the interaction vertices contain at most two spatiotemporal derivatives of the fields, we investigate the consistent interactions between a single massless tensor field with the mixed symmetry $(3, 1)$ and one massless tensor field with the mixed symmetry $(2, 2)$. The computations are done with the help of the deformation theory based on a cohomological approach, in the context of the antifield-BRST formalism. Our result is that dual linearized gravity in $D = 6$ gets coupled to a purely spin-two field with the mixed symmetry of the

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Riemann tensor such that both the gauge transformations and first-order reducibility relations in the $(3,1)$ sector are changed, but not the gauge algebra.

PACS number: 11.10.Ef

1 Introduction

Tensor fields in “exotic” representations of the Lorentz group, characterized by a mixed Young symmetry type [1, 2, 3, 4, 5, 6, 7], held the attention lately on some important issues, like the dual formulation of field theories of spin two or higher [8, 9, 10, 11, 12, 13, 14], the impossibility of consistent cross-interactions in the dual formulation of linearized gravity [15], or a Lagrangian first-order approach [16, 17] to some classes of massless or partially massive mixed symmetry-type tensor gauge fields, suggestively resembling to the tetrad formalism of General Relativity. An important matter related to mixed symmetry-type tensor fields is the study of their consistent interactions, among themselves as well as with higher-spin gauge theories [18, 19, 20, 21]. The most efficient approach to this problem is the cohomological one, based on the deformation of the solution to the master equation [22]. Until recently, it was commonly believed that mixed symmetry type tensor fields are rather rigid under the introduction of consistent interactions. Nevertheless, it has been proved that some classes of massless tensor fields with the mixed symmetry $(k,1)$, which are particularly important in view of their duality to linearized gravity in $D = k + 3$, allow nontrivial couplings: to a vector field for $k = 3$ [23], to an arbitrary p -form again for $k = 3$ [24], and to a topological BF model for $k = 2$ [25].

The purpose of this paper is to investigate the consistent interactions between a single massless tensor field with the mixed symmetry $(3,1)$ and one massless tensor field with the mixed symmetry $(2,2)$. Our analysis relies on the deformation of the solution to the master equation by means of cohomological techniques with the help of the local BRST cohomology, whose component in the $(3,1)$ sector has been reported in detail in [26] and in the $(2,2)$ sector has been investigated in [27, 28]. Apart from the duality of the massless tensor field with the mixed symmetry $(3,1)$ to the Pauli–Fierz theory (linearized limit of Einstein–Hilbert gravity) in $D = 6$ dimensions, it is interesting to mention the developments concerning the dual formulations of linearized gravity from the perspective of M -theory [29, 30, 31]. On

the other hand, the massless tensor field with the mixed symmetry $(2,2)$ displays all the algebraic properties of the Riemann tensor, describes purely spin-two particles, and also provides a dual formulation of linearized gravity in $D = 5$. Actually, there is a revived interest in the construction of dual gravity theories, which led to several new results, viz. a dual formulation of linearized gravity in first order tetrad formalism in arbitrary dimensions within the path integral framework [32] or a reformulation of non-linear Einstein gravity in terms of the dual graviton together with the ordinary metric and a shift gauge field [33].

Under the hypotheses of analyticity, locality, Lorentz covariance, and Poincaré invariance of the deformations, combined with the requirement that the interaction vertices contain at most two spatiotemporal derivatives of the fields, we prove that there exists a case where the deformation of the solution to the master equation provides non-trivial cross-couplings. This case corresponds to a six-dimensional spacetime and is described by a deformed solution that stops at order two in the coupling constant. In this way we establish a new result, namely that dual linearized gravity in $D = 6$ gets coupled to a purely spin-two field with the mixed symmetry of the Riemann tensor. The interacting Lagrangian action contains only mixing-component terms of order one and two in the coupling constant. This is the first time when both the gauge transformations and first-order reducibility functions of the tensor field $(3,1)$ are modified at order one in the coupling constant with terms characteristic to the $(2,2)$ sector. On the contrary, the tensor field with the mixed symmetry $(2,2)$ remains rigid at the level of both gauge transformations and reducibility functions. The gauge algebra and the reducibility structure of order two are not modified during the deformation procedure, being the same like in the case of the starting free action. It is interesting to note that if we require the PT invariance of the deformed theory, then no interactions occur.

2 Free model. BRST symmetry

We begin with the free Lagrangian action

$$S_0 [t_{\lambda\mu\nu|\alpha}, r_{\mu\nu|\alpha\beta}] = S_0 [t_{\lambda\mu\nu|\alpha}] + S_0 [r_{\mu\nu|\alpha\beta}], \quad (1)$$

in $D \geq 5$ spatiotemporal dimensions, where

$$\begin{aligned}
S_0 [t_{\lambda\mu\nu|\alpha}] &= \int d^D x \left\{ \frac{1}{2} [(\partial^\rho t^{\lambda\mu\nu|\alpha}) (\partial_\rho t_{\lambda\mu\nu|\alpha}) - (\partial_\alpha t^{\lambda\mu\nu|\alpha}) (\partial^\beta t_{\lambda\mu\nu|\beta})] \right. \\
&\quad - \frac{3}{2} [(\partial_\lambda t^{\lambda\mu\nu|\alpha}) (\partial^\rho t_{\rho\mu\nu|\alpha}) + (\partial^\rho t^{\lambda\mu}) (\partial_\rho t_{\lambda\mu})] \\
&\quad \left. + 3 (\partial_\alpha t^{\lambda\mu\nu|\alpha}) (\partial_\lambda t_{\mu\nu}) + 3 (\partial_\rho t^{\rho\mu}) (\partial^\lambda t_{\lambda\mu}) \right\} \quad (2)
\end{aligned}$$

and

$$\begin{aligned}
S_0 [r_{\mu\nu|\alpha\beta}] &= \int d^D x \left[\frac{1}{8} (\partial^\lambda r^{\mu\nu|\alpha\beta}) (\partial_\lambda r_{\mu\nu|\alpha\beta}) - \frac{1}{2} (\partial_\mu r^{\mu\nu|\alpha\beta}) (\partial^\lambda r_{\lambda\nu|\alpha\beta}) \right. \\
&\quad - (\partial_\mu r^{\mu\nu|\alpha\beta}) (\partial_\beta r_{\nu\alpha}) - \frac{1}{2} (\partial^\lambda r^{\nu\beta}) (\partial_\lambda r_{\nu\beta}) \\
&\quad \left. + (\partial_\nu r^{\nu\beta}) (\partial^\lambda r_{\lambda\beta}) - \frac{1}{2} (\partial_\nu r^{\nu\beta}) (\partial_\beta r) + \frac{1}{8} (\partial^\lambda r) (\partial_\lambda r) \right]. \quad (3)
\end{aligned}$$

Everywhere in this paper we employ the flat Minkowski metric of ‘mostly plus’ signature $\sigma^{\mu\nu} = \sigma_{\mu\nu} = (- + + + + \dots)$. The massless tensor field $t_{\lambda\mu\nu|\alpha}$ has the mixed symmetry $(3, 1)$, and hence transforms according to an irreducible representation of $GL(D, \mathbb{R})$ corresponding to a 4-cell Young diagram with two columns and three rows. It is thus completely antisymmetric in its first three indices and satisfies the identity $t_{[\lambda\mu\nu|\alpha]} \equiv 0$. Here and in the sequel the notation $[\lambda \dots \alpha]$ signifies complete antisymmetry with respect to the (Lorentz) indices between brackets, with the conventions that the minimum number of terms is always used and the result is never divided by the number of terms. The trace of $t_{\lambda\mu\nu|\alpha}$ is defined by $t_{\lambda\mu} = \sigma^{\nu\alpha} t_{\lambda\mu\nu|\alpha}$ and it is obviously an antisymmetric tensor. The massless tensor field $r_{\mu\nu|\alpha\beta}$ of degree four has the mixed symmetry of the linearized Riemann tensor, and hence transforms according to an irreducible representation of $GL(D, \mathbb{R})$, corresponding to the rectangular Young diagram $(2, 2)$ with two columns and two rows. Thus, it is separately antisymmetric in the pairs $\{\mu, \nu\}$ and $\{\alpha, \beta\}$, is symmetric under the interchange of these pairs ($\{\mu, \nu\} \longleftrightarrow \{\alpha, \beta\}$), and satisfies the identity $r_{[\mu\nu|\alpha]\beta} \equiv 0$ associated with the above diagram. The notation $r_{\nu\beta}$ signifies the trace of the original tensor field, $r_{\nu\beta} = \sigma^{\mu\alpha} r_{\mu\nu|\alpha\beta}$, which is symmetric, $r_{\nu\beta} = r_{\beta\nu}$, while r denotes its double trace, $r = \sigma^{\nu\beta} r_{\nu\beta} \equiv r^{\mu\nu}_{|\mu\nu}$, which is a scalar.

A generating set of gauge transformations for action (1) can be taken of the form

$$\delta_{\epsilon, \chi} t_{\lambda\mu\nu|\alpha} = 3\partial_\alpha \epsilon_{\lambda\mu\nu} + \partial_{[\lambda} \epsilon_{\mu\nu]\alpha} + \partial_{[\lambda} \chi_{\mu\nu]|\alpha}, \quad (4)$$

$$\delta_\xi r_{\mu\nu|\alpha\beta} = \partial_\mu \xi_{\alpha\beta|\nu} - \partial_\nu \xi_{\alpha\beta|\mu} + \partial_\alpha \xi_{\mu\nu|\beta} - \partial_\beta \xi_{\mu\nu|\alpha}. \quad (5)$$

The gauge parameters $\epsilon_{\lambda\mu\nu}$ determine a completely antisymmetric tensor, while the gauge parameters $\chi_{\mu\nu|\alpha}$ and $\xi_{\mu\nu|\alpha}$ display the mixed symmetry $(2,1)$, such that they are antisymmetric in the first two indices and satisfy the identities $\chi_{[\mu\nu|\alpha]} \equiv 0$ and $\xi_{[\mu\nu|\alpha]} \equiv 0$. The generating set of gauge transformations (4)–(5) is off-shell, second-stage reducible, the accompanying gauge algebra being obviously Abelian. More precisely, the gauge transformations (4) are off-shell second-stage reducible because: 1. if in (4)–(5) we make the transformations

$$\epsilon_{\mu\nu\alpha} \rightarrow \epsilon_{\mu\nu\alpha}^{(\omega)} = -\frac{1}{2}\partial_{[\mu}\omega_{\nu]\alpha}, \quad (6)$$

$$\chi_{\mu\nu|\alpha} \rightarrow \chi_{\mu\nu|\alpha}^{(\omega,\psi)} = 2\partial_\alpha\omega_{\mu\nu} - \partial_{[\mu}\omega_{\nu]\alpha} + \partial_{[\mu}\psi_{\nu]\alpha}, \quad (7)$$

$$\xi_{\mu\nu|\alpha} \rightarrow \xi_{\mu\nu|\alpha}^{(\varphi)} = 2\partial_\alpha\varphi_{\mu\nu} - \partial_{[\mu}\varphi_{\nu]\alpha}, \quad (8)$$

with $\omega_{\nu\alpha}$ and $\varphi_{\mu\nu}$ antisymmetric and $\psi_{\nu\alpha}$ symmetric (but otherwise arbitrary), then the gauge variations of both tensor fields identically vanish $\delta_{\epsilon^{(\omega)}, \chi^{(\omega,\psi)}} t_{\lambda\mu\nu|\alpha} \equiv 0$, $\delta_{\xi^{(\varphi)}} r_{\mu\nu|\alpha\beta} \equiv 0$; 2. there is no non-vanishing local transformation of $\varphi_{\mu\nu}$ that annihilates $\xi_{\mu\nu|\alpha}^{(\varphi)}$ of the form (8), and hence no further local reducibility identity related with (5); 3. if in (6)–(7) we perform the changes

$$\omega_{\nu\alpha} \rightarrow \omega_{\nu\alpha}^{(\theta)} = \partial_{[\nu}\theta_{\alpha]}, \quad \psi_{\nu\alpha} \rightarrow \psi_{\nu\alpha}^{(\theta)} = -3\partial_{(\nu}\theta_{\alpha)}, \quad (9)$$

with θ_α an arbitrary vector field, where $(\nu\alpha \cdots)$ signifies symmetrization with respect to the indices between parentheses without normalization factors, then the transformed gauge parameters (6)–(7) identically vanish $\epsilon_{\mu\nu\alpha}^{(\omega^{(\theta)})} \equiv 0$, $\chi_{\mu\nu|\alpha}^{(\omega^{(\theta)}, \psi^{(\theta)})} \equiv 0$; 4. there is no non-vanishing local transformation of θ_ν that simultaneously annihilates $\omega_{\nu\alpha}^{(\theta)}$ and $\psi_{\nu\alpha}^{(\theta)}$ of the form (9), and hence no further local reducibility identity related to (4).

The construction of the antifield-BRST symmetry for this free theory debuts with the identification of the algebra on which the BRST differential s acts. The generators of the BRST algebra are of two kinds: fields/ghosts and antifields. The ghost spectrum for the model under study comprises the fermionic ghosts $\{\eta_{\lambda\mu\nu}, \mathcal{G}_{\mu\nu|\alpha}, \mathcal{C}_{\mu\nu|\alpha}\}$ associated with the gauge parameters $\{\epsilon_{\lambda\mu\nu}, \chi_{\mu\nu|\alpha}, \xi_{\mu\nu|\alpha}\}$ from (4)–(5), the bosonic ghosts for ghosts $\{C_{\mu\nu}, G_{\nu\alpha}, \mathcal{C}_{\mu\nu}\}$ due to the first-stage reducibility parameters $\{\omega_{\mu\nu}, \psi_{\nu\alpha}, \varphi_{\mu\nu}\}$ in (6)–(8), and

also the fermionic ghost for ghost for ghost C_ν corresponding to the second-stage reducibility parameter θ_ν in (9). In order to make compatible the behavior of the gauge and reducibility parameters with that of the accompanying ghosts, we ask that $\eta_{\lambda\mu\nu}$, $C_{\mu\nu}$, and $\mathcal{C}_{\mu\nu}$ are completely antisymmetric, $\mathcal{G}_{\mu\nu|\alpha}$ and $\mathcal{C}_{\mu\nu|\alpha}$ obey the analogue of the properties fulfilled by the gauge parameters $\chi_{\mu\nu|\alpha}$ and $\xi_{\mu\nu|\alpha}$, while $G_{\nu\alpha}$ is symmetric. The antifield spectrum is organized into the antifields $\{t^{*\lambda\mu\nu|\alpha}, r^{*\mu\nu|\alpha\beta}\}$ of the original tensor fields, together with those of the ghosts, $\{\eta^{*\lambda\mu\nu}, \mathcal{G}^{*\mu\nu|\alpha}, \mathcal{C}^{*\mu\nu|\alpha}\}$, $\{C^{*\mu\nu}, G^{*\nu\alpha}, \mathcal{C}^{*\mu\nu}\}$ and respectively $C^{*\nu}$, of statistics opposite to that of the associated field/ghost. It is understood that $t^{*\lambda\mu\nu|\alpha}$ and $r^{*\mu\nu|\alpha\beta}$ exhibit the same mixed-symmetry properties like $t_{\lambda\mu\nu|\alpha}$ and $r_{\mu\nu|\alpha\beta}$ and similarly with respect to $\eta^{*\lambda\mu\nu}$, $\mathcal{G}^{*\mu\nu|\alpha}$, $\mathcal{C}^{*\mu\nu|\alpha}$, $G^{*\nu\alpha}$, $C^{*\mu\nu}$, and $\mathcal{C}^{*\mu\nu}$. For subsequent purposes, we denote the trace of $t^{*\lambda\mu\nu|\alpha}$ by $t^{*\lambda\mu}$, being understood that it is antisymmetric. Similarly, the notation $r^{*\nu\beta}$ signifies the trace of $r^{*\mu\nu|\alpha\beta}$, $r^{*\nu\beta} = \sigma_{\mu\alpha} r^{*\mu\nu|\alpha\beta}$, which is symmetric, $r^{*\nu\beta} = r^{*\beta\nu}$, while r^* denotes its double trace, $r^* = \sigma^{*\nu\beta} r_{\nu\beta} \equiv r_{\mu\nu}^{*\mu\nu}$, which is a scalar.

Since both the gauge generators and reducibility functions for this model are field-independent, it follows that the BRST differential s simply reduces to

$$s = \delta + \gamma, \quad (10)$$

where δ represents the Koszul–Tate differential, graded by the antighost number agh ($\text{agh}(\delta) = -1$), and γ stands for the exterior derivative along the gauge orbits, whose degree is named pure ghost number pgh ($\text{pgh}(\gamma) = 1$). These two degrees do not interfere ($\text{agh}(\gamma) = 0$, $\text{pgh}(\delta) = 0$). The overall degree that grades the BRST complex is known as the ghost number (gh) and is defined like the difference between the pure ghost number and the antighost number, such that $\text{gh}(s) = \text{gh}(\delta) = \text{gh}(\gamma) = 1$. According to the standard rules of the BRST method, the corresponding degrees of the generators from the BRST complex are valued like

$$\text{pgh}(t_{\lambda\mu\nu|\alpha}) = 0 = \text{pgh}(r_{\mu\nu|\alpha\beta}), \quad (11)$$

$$\text{pgh}(\eta_{\lambda\mu\nu}) = \text{pgh}(\mathcal{G}_{\mu\nu|\alpha}) = \text{pgh}(\mathcal{C}_{\mu\nu|\alpha}) = 1, \quad (12)$$

$$\text{pgh}(C_{\mu\nu}) = \text{pgh}(\mathcal{C}_{\mu\nu}) = \text{pgh}(G_{\nu\alpha}) = 2, \quad \text{pgh}(C_\nu) = 3, \quad (13)$$

$$\text{pgh}(t^{*\lambda\mu\nu|\alpha}) = 0 = \text{pgh}(r^{*\mu\nu|\alpha\beta}), \quad (14)$$

$$\text{pgh}(\eta^{*\lambda\mu\nu}) = \text{pgh}(\mathcal{G}^{*\mu\nu|\alpha}) = \text{pgh}(\mathcal{C}^{*\mu\nu|\alpha}) = 0, \quad (15)$$

$$\text{pgh}(C^{*\mu\nu}) = \text{pgh}(\mathcal{C}^{*\mu\nu}) = \text{pgh}(G^{*\nu\alpha}) = \text{pgh}(C^{*\nu}) = 0, \quad (16)$$

$$\text{agh}(t_{\lambda\mu\nu|\alpha}) = 0 = \text{agh}(r_{\mu\nu|\alpha\beta}), \quad (17)$$

$$\text{agh}(\eta_{\lambda\mu\nu}) = \text{agh}(\mathcal{G}_{\mu\nu|\alpha}) = \text{agh}(\mathcal{C}_{\mu\nu|\alpha}) = 0, \quad (18)$$

$$\text{agh}(C_{\mu\nu}) = \text{agh}(\mathcal{C}_{\mu\nu}) = \text{agh}(G_{\nu\alpha}) = \text{agh}(C_\nu) = 0, \quad (19)$$

$$\text{agh}(t^{*\lambda\mu\nu|\alpha}) = 1 = \text{agh}(r^{*\mu\nu|\alpha\beta}), \quad (20)$$

$$\text{agh}(\eta^{*\lambda\mu\nu}) = \text{agh}(\mathcal{G}^{*\mu\nu|\alpha}) = \text{agh}(\mathcal{C}^{*\mu\nu|\alpha}) = 2, \quad (21)$$

$$\text{agh}(C^{*\mu\nu}) = \text{agh}(\mathcal{C}^{*\mu\nu}) = \text{agh}(G^{*\nu\alpha}) = 3, \quad \text{agh}(C^{*\nu}) = 4. \quad (22)$$

Actually, (10) is a decomposition of the BRST differential according to the antighost number and it shows that s contains only components of antighost number equal to minus one and zero. The Koszul–Tate differential is imposed to realize a homological resolution of the algebra of smooth functions defined on the stationary surface of field equations, while the exterior longitudinal derivative is related to the gauge symmetries (see relations (4)–(5)) of action (1) through its cohomology at pure ghost number zero computed in the cohomology of δ , which is required to be the algebra of physical observables for the free model under consideration. The actions of δ and γ on the generators from the BRST complex, which enforce all the above mentioned properties, are given by

$$\gamma t_{\lambda\mu\nu|\alpha} = 3\partial_\alpha \eta_{\lambda\mu\nu} + \partial_{[\lambda} \eta_{\mu\nu]\alpha} + \partial_{[\lambda} \mathcal{G}_{\mu\nu]|\alpha}, \quad (23)$$

$$\gamma r_{\mu\nu|\alpha\beta} = \partial_\mu \mathcal{C}_{\alpha\beta|\nu} - \partial_\nu \mathcal{C}_{\alpha\beta|\mu} + \partial_\alpha \mathcal{C}_{\mu\nu|\beta} - \partial_\beta \mathcal{C}_{\mu\nu|\alpha}, \quad (24)$$

$$\gamma \eta_{\lambda\mu\nu} = -\frac{1}{2} \partial_{[\lambda} C_{\mu\nu]}, \quad \gamma \mathcal{C}_{\mu\nu|\alpha} = 2\partial_\alpha \mathcal{C}_{\mu\nu} - \partial_{[\mu} \mathcal{C}_{\nu]\alpha}, \quad (25)$$

$$\gamma \mathcal{G}_{\mu\nu|\alpha} = 2\partial_{[\mu} C_{\nu]\alpha} - 3\partial_{[\mu} C_{\nu]\alpha} + \partial_{[\mu} G_{\nu]\alpha}, \quad (26)$$

$$\gamma C_{\mu\nu} = \partial_{[\mu} C_{\nu]}, \quad \gamma G_{\nu\alpha} = -3\partial_{(\nu} C_{\alpha)}, \quad \gamma \mathcal{C}_{\mu\nu} = \gamma C_\nu = 0, \quad (27)$$

$$\gamma t^{*\lambda\mu\nu|\alpha} = \gamma r^{*\mu\nu|\alpha\beta} = \gamma \eta^{*\lambda\mu\nu} = \gamma \mathcal{G}^{*\mu\nu|\alpha} = \gamma \mathcal{C}^{*\mu\nu|\alpha} = 0, \quad (28)$$

$$\gamma C^{*\mu\nu} = \gamma G^{*\nu\alpha} = \gamma \mathcal{C}^{*\mu\nu} = \gamma C^{*\nu} = 0, \quad (29)$$

$$\delta t_{\lambda\mu\nu|\alpha} = \delta r_{\mu\nu|\alpha\beta} = \delta \eta_{\lambda\mu\nu} = \delta \mathcal{G}_{\mu\nu|\alpha} = \delta \mathcal{C}_{\mu\nu|\alpha} = 0, \quad (30)$$

$$\delta C_{\mu\nu} = \delta G_{\nu\alpha} = \delta \mathcal{C}_{\mu\nu} = \delta C_\nu = 0, \quad (31)$$

$$\delta t^{*\lambda\mu\nu|\alpha} = -\frac{\delta S_0[t_{\lambda\mu\nu|\alpha}]}{\delta t_{\lambda\mu\nu|\alpha}}, \quad \delta r^{*\mu\nu|\alpha\beta} = -\frac{\delta S_0[r_{\mu\nu|\alpha\beta}]}{\delta r_{\mu\nu|\alpha\beta}}, \quad (32)$$

$$\delta \eta^{*\lambda\mu\nu} = -4\partial_\alpha t^{*\lambda\mu\nu|\alpha}, \quad \delta \mathcal{G}^{*\mu\nu|\alpha} = -\partial_\lambda (3t^{*\lambda\mu\nu|\alpha} - t^{*\mu\nu\alpha|\lambda}), \quad (33)$$

$$\delta \mathcal{C}^{*\alpha\beta|\nu} = -4\partial_\mu r^{*\mu\nu|\alpha\beta}, \quad \delta C^{*\mu\nu} = 3\partial_\lambda \left(\mathcal{G}^{*\mu\nu|\lambda} - \frac{1}{2} \eta^{*\lambda\mu\nu} \right), \quad (34)$$

$$\delta G^{*\nu\alpha} = \partial_\mu \mathcal{G}^{*\mu(\nu|\alpha)}, \quad \delta \mathcal{C}^{*\mu\nu} = 3\partial_\alpha \mathcal{C}^{*\mu\nu|\alpha}, \quad (35)$$

$$\delta C^{*\nu} = 6\partial_\mu \left(G^{*\mu\nu} - \frac{1}{3} C^{*\mu\nu} \right). \quad (36)$$

By convention, we take δ and γ to act like right derivations. We note that the action of the Koszul–Tate differential on the antifields with the antighost number equal to two and respectively three gains a simpler expression if we perform the changes of variables

$$\mathcal{G}'^{*\mu\nu|\alpha} = \mathcal{G}^{*\mu\nu|\alpha} + \frac{1}{4} \eta^{*\mu\nu\alpha}, \quad G'^{*\nu\alpha} = G^{*\nu\alpha} - \frac{1}{3} C^{*\nu\alpha}. \quad (37)$$

The antifield $\mathcal{G}'^{*\mu\nu|\alpha}$ is still antisymmetric in its first two indices, but does not fulfill any longer the identity $\mathcal{G}'^{*[\mu\nu|\alpha]} \equiv 0$, and $G'^{*\nu\alpha}$ has no definite symmetry or antisymmetry properties. With the help of relations (33)–(36), we find that δ acts on the transformed antifields through the relations

$$\delta \mathcal{G}'^{*\mu\nu|\alpha} = -3\partial_\lambda t^{*\lambda\mu\nu|\alpha}, \quad \delta G'^{*\nu\alpha} = 2\partial_\mu \mathcal{G}'^{*\mu\nu|\alpha}, \quad \delta C^{*\nu} = 6\partial_\mu G'^{*\mu\nu}. \quad (38)$$

The same observation is valid with respect to γ if we make the changes

$$\mathcal{G}'_{\mu\nu|\alpha} = \mathcal{G}_{\mu\nu|\alpha} + 4\eta_{\mu\nu\alpha}, \quad G'_{\nu\alpha} = G_{\nu\alpha} - 3C_{\nu\alpha}, \quad (39)$$

in terms of which we can write

$$\gamma t_{\lambda\mu\nu|\alpha} = -\frac{1}{4} \partial_{[\lambda} \mathcal{G}'_{\mu\nu|\alpha]} + \partial_{[\lambda} \mathcal{G}'_{\mu\nu]|\alpha}, \quad \gamma \mathcal{G}'_{\mu\nu|\alpha} = \partial_{[\mu} G'_{\nu]\alpha}, \quad \gamma G'_{\nu\alpha} = -6\partial_\nu C_\alpha. \quad (40)$$

Again, $\mathcal{G}'_{\mu\nu|\alpha}$ is antisymmetric in its first two indices, but does not satisfy the identity $\mathcal{G}'_{[\mu\nu|\alpha]} \equiv 0$, while $G'_{\nu\alpha}$ has no definite symmetry or antisymmetry. We have deliberately chosen the same notations for the transformed variables (37) and (39) since they actually form pairs that are conjugated in the antibracket.

The Lagrangian BRST differential admits a canonical action in a structure named antibracket and defined by decreeing the fields/ghosts conjugated with the corresponding antifields, $s \cdot = (\cdot, S)$, where (\cdot, \cdot) signifies the antibracket and S denotes the canonical generator of the BRST symmetry. It is a bosonic functional of ghost number zero, involving both field/ghost and antifield spectra, that obeys the master equation

$$(S, S) = 0. \quad (41)$$

The master equation is equivalent with the second-order nilpotency of s , where its solution S encodes the entire gauge structure of the associated theory. Taking into account formulae (23)–(36) as well as the standard actions of δ and γ in canonical form, we find that the complete solution to the master equation for the free model under study is given by

$$\begin{aligned}
S = & S_0 [t_{\lambda\mu\nu|\alpha}, r_{\mu\nu|\alpha\beta}] + \int d^D x [t^{*\lambda\mu\nu|\alpha} (3\partial_\alpha \eta_{\lambda\mu\nu} + \partial_{[\lambda} \eta_{\mu\nu]\alpha} + \partial_{[\lambda} \mathcal{G}_{\mu\nu]|\alpha}) \\
& - \frac{1}{2} \eta^{*\mu\nu\alpha} \partial_{[\mu} C_{\nu]\alpha} + \mathcal{G}^{*\mu\nu|\alpha} (2\partial_\alpha C_{\mu\nu} - \partial_{[\mu} C_{\nu]\alpha} + \partial_{[\mu} G_{\nu]\alpha}) \\
& + C^{*\mu\nu} \partial_{[\mu} C_{\nu]} - 3G^{*\nu\alpha} \partial_{(\nu} C_{\alpha)} + r^{*\mu\nu|\alpha\beta} (\partial_\mu \mathcal{C}_{\alpha\beta|\nu} - \partial_\nu \mathcal{C}_{\alpha\beta|\mu} \\
& + \partial_\alpha \mathcal{C}_{\mu\nu|\beta} - \partial_\beta \mathcal{C}_{\mu\nu|\alpha}) + \mathcal{C}^{*\mu\nu|\alpha} (2\partial_\alpha \mathcal{C}_{\mu\nu} - \partial_{[\mu} \mathcal{C}_{\nu]\alpha})], \tag{42}
\end{aligned}$$

such that it contains pieces with the antighost number ranging from zero to three.

3 Brief review of the deformation procedure

There are three main types of consistent interactions that can be added to a given gauge theory: *(i)* the first type deforms only the Lagrangian action, but not its gauge transformations, *(ii)* the second kind modifies both the action and its transformations, but not the gauge algebra, and *(iii)* the third, and certainly most interesting category, changes everything, namely, the action, its gauge symmetries and the accompanying algebra.

The reformulation of the problem of consistent deformations of a given action and of its gauge symmetries in the antifield-BRST setting is based on the observation that if a deformation of the classical theory can be consistently constructed, then the solution to the master equation for the initial theory can be deformed into the solution of the master equation for the interacting theory

$$\bar{S} = S + gS_1 + g^2S_2 + O(g^3), \quad \varepsilon(\bar{S}) = 0, \quad \text{gh}(\bar{S}) = 0, \tag{43}$$

such that

$$(\bar{S}, \bar{S}) = 0. \tag{44}$$

Here and in the sequel $\varepsilon(F)$ denotes the Grassmann parity of F . The projection of (43) on the various powers of the coupling constant induces the

following tower of equations:

$$g^0 : (S, S) = 0, \quad (45)$$

$$g^1 : (S_1, S) = 0, \quad (46)$$

$$g^2 : \frac{1}{2} (S_1, S_1) + (S_2, S) = 0, \quad (47)$$

$$g^3 : (S_1, S_2) + (S_3, S) = 0, \quad (48)$$

$$g^4 : \frac{1}{2} (S_2, S_2) + (S_1, S_3) + (S_4, S) = 0, \quad (49)$$

\vdots

The first equation is satisfied by hypothesis. The second one governs the first-order deformation of the solution to the master equation, S_1 , and it expresses the fact that S_1 is a BRST co-cycle, $sS_1 = 0$, and hence it exists and is local. The remaining equations are responsible for the higher-order deformations of the solution to the master equation. No obstructions arise in finding solutions to them as long as no further restrictions, such as spatiotemporal locality, are imposed. Obviously, only non-trivial first-order deformations should be considered, since trivial ones ($S_1 = sB$) lead to trivial deformations of the initial theory, and can be eliminated by convenient redefinitions of the fields. Ignoring the trivial deformations, it follows that S_1 is a non-trivial BRST-observable, $S_1 \in H^0(s)$ (where $H^0(s)$ denotes the cohomology space of the BRST differential at ghost number zero). Once the deformation equations ((46)–(49), etc.) have been solved by means of specific cohomological techniques, from the consistent non-trivial deformed solution to the master equation one can extract all the information on the gauge structure of the resulting interacting theory.

4 Main result

The aim of this paper is to investigate the consistent interactions that can be added to action (1) without modifying either the field/ghost/antifield spectrum or the number of independent gauge symmetries. This matter is addressed in the context of the antifield-BRST deformation procedure described in the above and relies on computing the solutions to Eqs. (46)–(49), etc., from the cohomology of the BRST differential. For obvious reasons, we consider only analytical, local, and manifestly covariant deformations and,

meanwhile, restrict to Poincaré-invariant quantities, i.e. we do not allow explicit dependence on the spatiotemporal coordinates. The analyticity of deformations refers to the fact that the deformed solution to the master equation, (43), is analytical in the coupling constant g and reduces to the original solution (42) in the free limit ($g = 0$). Moreover, we ask that the deformed gauge theory preserves the Cauchy order of the uncoupled model, which enforces the requirement that the interacting Lagrangian is of maximum order equal to two in the spatiotemporal derivatives of the fields at each order in the coupling constant. Here, we present the main result without insisting on the cohomology tools required by the technique of consistent deformations. All the cohomological proofs that lead to the main result are given in the next section.

The fully deformed solution to the master equation (44) ends at order two in the coupling constant and is given by

$$\begin{aligned}\bar{S} = & S + g \int d^6x \left[\varepsilon_{\mu\nu\alpha\lambda\beta\gamma} \eta^{*\mu\nu\alpha} \partial^\lambda \mathcal{C}^{\beta\gamma} - 2t^{*\lambda\mu\nu|\alpha} \varepsilon_{\lambda\mu\nu\rho\beta\gamma} \left(\partial^\rho \mathcal{C}^{\beta\gamma|}{}_\alpha \right. \right. \\ & \left. \left. - \frac{1}{4} \delta^\gamma_\alpha \partial^{[\rho} \mathcal{C}^{\beta\tau]}{}_\tau \right) - 2t_{\lambda\mu\nu|\rho} \varepsilon^{\lambda\mu\nu\alpha\beta\gamma} \left(\partial_\sigma \partial_\alpha r_{\beta\gamma|}{}^{\sigma\rho} - \frac{1}{2} \delta^\rho_\gamma \partial^\tau \partial_\alpha r_{\beta\tau} \right) + r \right] \\ & - g^2 \int d^6x \left(5r^{\lambda\rho|[\alpha\beta,\gamma]} r_{\lambda\rho|[\alpha\beta,\gamma]} - 6r_{\lambda\rho|}{}^{[\alpha\beta,\rho]} r^{\lambda\sigma|}{}_{[\alpha\beta,\sigma]} \right),\end{aligned}\quad (50)$$

where

$$r^{\lambda\rho|[\alpha\beta,\gamma]} = \partial^\gamma r^{\lambda\rho|\alpha\beta} + \partial^\beta r^{\lambda\rho|\gamma\alpha} + \partial^\alpha r^{\lambda\rho|\beta\gamma}. \quad (51)$$

We recall that r denotes the double trace of $r_{\mu\nu|\alpha\beta}$. We observe that this solution ‘lives’ in a six-dimensional spacetime. From (50) we read all the information on the gauge structure of the coupled theory. The terms of antighost number zero in (50) provide the Lagrangian action. They can be organized as

$$\begin{aligned}\bar{S}_0 [t_{\lambda\mu\nu|\alpha}, r_{\mu\nu|\alpha\beta}] = & S_0 [t_{\lambda\mu\nu|\alpha}, r_{\mu\nu|\alpha\beta}] + g \int d^6x \left[r \right. \\ & \left. - 2t_{\lambda\mu\nu|\rho} \varepsilon^{\lambda\mu\nu\alpha\beta\gamma} \left(\partial_\sigma \partial_\alpha r_{\beta\gamma|}{}^{\sigma\rho} - \frac{1}{2} \delta^\rho_\gamma \partial^\tau \partial_\alpha r_{\beta\tau} \right) \right. \\ & \left. - g \left(5r^{\lambda\rho|[\alpha\beta,\gamma]} r_{\lambda\rho|[\alpha\beta,\gamma]} - 6r_{\lambda\rho|}{}^{[\alpha\beta,\rho]} r^{\lambda\sigma|}{}_{[\alpha\beta,\sigma]} \right) \right],\end{aligned}\quad (52)$$

where $S_0 [t_{\lambda\mu\nu|\alpha}, r_{\mu\nu|\alpha\beta}]$ is the Lagrangian action appearing in (1) in $D = 6$. We observe that action (52) contains only mixing-component terms of order

one and two in the coupling constant. The piece of antighost number one appearing in (50) gives the deformed gauge transformations in the form

$$\begin{aligned}\bar{\delta}_{\epsilon,\chi,\xi} t_{\lambda\mu\nu|\alpha} &= 3\partial_\alpha \epsilon_{\lambda\mu\nu} + \partial_{[\lambda} \epsilon_{\mu\nu]\alpha} + \partial_{[\lambda} \chi_{\mu\nu]|\alpha} \\ &\quad - 2g\varepsilon_{\lambda\mu\nu\rho\beta\gamma} \left(\partial^\rho \xi^{\beta\gamma|}_{\alpha} - \frac{1}{4} \delta^\gamma_\alpha \partial^{[\rho} \xi^{\beta\tau]|\tau} \right),\end{aligned}\quad (53)$$

$$\bar{\delta}_\xi r_{\mu\nu|\alpha\beta} = \partial_\mu \xi_{\alpha\beta|\nu} - \partial_\nu \xi_{\alpha\beta|\mu} + \partial_\alpha \xi_{\mu\nu|\beta} - \partial_\beta \xi_{\mu\nu|\alpha} = \delta_\xi r_{\mu\nu|\alpha\beta}. \quad (54)$$

It is interesting to note that only the gauge transformations of the tensor field (3,1) are modified during the deformation process. This is enforced at order one in the coupling constant by a term linear in the first-order derivatives of the gauge parameters from the (2,2) sector. From the terms of antighost number equal to two present in (50) we learn that only the first-order reducibility functions (see (6)) are modified at order one in the coupling constant, the others coinciding with the original ones

$$\epsilon_{\mu\nu\alpha} \rightarrow \bar{\epsilon}_{\mu\nu\alpha}^{(\omega,\varphi)} = -\frac{1}{2} \partial_{[\mu} \omega_{\nu]\alpha} + g\varepsilon_{\mu\nu\alpha\lambda\beta\gamma} \partial^\lambda \varphi^{\beta\gamma}, \quad (55)$$

$$\chi_{\mu\nu|\alpha} \rightarrow \bar{\chi}_{\mu\nu|\alpha}^{(\omega,\psi)} = 2\partial_\alpha \omega_{\mu\nu} - \partial_{[\mu} \omega_{\nu]\alpha} + \partial_{[\mu} \psi_{\nu]\alpha}, \quad (56)$$

$$\xi_{\mu\nu|\alpha} \rightarrow \bar{\xi}_{\mu\nu|\alpha}^{(\varphi)} = 2\partial_\alpha \varphi_{\mu\nu} - \partial_{[\mu} \varphi_{\nu]\alpha}. \quad (57)$$

Consequently, the first-order reducibility relations for $t_{\lambda\mu\nu|\alpha}$ become

$$\bar{\delta}_{\bar{\epsilon}^{(\omega,\varphi)}, \bar{\chi}^{(\omega,\psi)}, \bar{\xi}^{(\varphi)}} t_{\lambda\mu\nu|\alpha} \equiv 0, \quad (58)$$

while those for $r_{\mu\nu|\alpha\beta}$ are not changed with respect to the free theory. Since there are no other terms of antighost number two in (50), it follows that the gauge algebra of the coupled model is unchanged by the deformation procedure, being the same Abelian one like for the starting free theory. The structure of pieces with the antighost number equal to three from (50) implies that the second-order reducibility functions (9) remain the same, and hence the second-order reducibility relations are exactly the initial ones. It is easy to see from (52)–(57) that if we impose the PT-invariance at the level of the coupled model, then we obtain no interactions (we must set $g = 0$ in these formulae).

It is important to stress that the problem of obtaining consistent interactions strongly depends on the spatiotemporal dimension. For instance, if one starts with action (1) in $D > 6$, then one inexorably gets $\bar{S} = S + g \int d^D x \, r$, so no cross-interaction term can be added to either the original Lagrangian or its gauge transformations.

5 Proof of the main result

In the sequel we prove the main result, stated in the previous section.

5.1 First-order deformation

If we make the notation $S_1 = \int d^D x a$, with a a local function, then the local form of Eq. (46), which we have seen that controls the first-order deformation of the solution to the master equation, becomes

$$sa = \partial_\mu m^\mu, \quad \text{gh}(a) = 0, \quad \varepsilon(a) = 0, \quad (59)$$

for some local m^μ , and it shows that the non-integrated density of the first-order deformation pertains to the local cohomology of s at ghost number zero, $a \in H^0(s|d)$, where d denotes the exterior spatiotemporal differential. In order to analyze the above equation, we develop a according to the antighost number

$$a = \sum_{k=0}^I a_k, \quad \text{agh}(a_k) = k, \quad \text{gh}(a_k) = 0, \quad \varepsilon(a_k) = 0, \quad (60)$$

and assume, without loss of generality, that the above decomposition stops at some finite value of the antighost number, I . By taking into account the splitting (10) of the BRST differential, Eq. (59) becomes equivalent to a tower of local equations, corresponding to the different decreasing values of the antighost number

$$\gamma a_I = \partial_\mu m^{(I)\mu}, \quad (61)$$

$$\delta a_I + \gamma a_{I-1} = \partial_\mu m^{(I-1)\mu}, \quad (62)$$

$$\delta a_k + \gamma a_{k-1} = \partial_\mu m^{(k-1)\mu}, \quad I-1 \geq k \geq 1, \quad (63)$$

where $\left(m^{(k)\mu}\right)_{k=\overline{0}, I}$ are some local currents with $\text{agh}\left(m^{(k)\mu}\right) = k$. It can be proved that we can replace Eq. (61) at strictly positive antighost numbers with

$$\gamma a_I = 0, \quad \text{agh}(a_I) = I > 0. \quad (64)$$

The proof can be done like in the Appendix A, Corollary 1, from [26]. In conclusion, under the assumption that $I > 0$, the representative of highest

antighost number from the non-integrated density of the first-order deformation can always be taken to be γ -closed, such that Eq. (59) associated with the local form of the first-order deformation is completely equivalent to the tower of equations given by (62)–(63) and (64).

Before proceeding to the analysis of the solutions to the first-order deformation equation, let us briefly comment on the uniqueness and triviality of such solutions. Due to the second-order nilpotency of γ ($\gamma^2 = 0$), the solution to the top equation, (64), is clearly unique up to γ -exact contributions, $a_I \rightarrow a_I + \gamma b_I$. Meanwhile, if it turns out that a_I reduces to γ -exact terms only, $a_I = \gamma b_I$, then it can be made to vanish, $a_I = 0$. In other words, the non-triviality of the first-order deformation a is translated at its highest antighost number component into the requirement that $a_I \in H^I(\gamma)$, where $H^I(\gamma)$ denotes the cohomology of the exterior longitudinal derivative γ at pure ghost number equal to I . At the same time, the general condition on the non-integrated density of the first-order deformation to be in a non-trivial cohomological class of $H^0(s|d)$ shows on the one hand that the solution to Eq. (59) is unique up to s -exact pieces plus total derivatives and, on the other hand, that if the general solution to (59) is completely trivial, $a = sb + \partial_\mu n^\mu$, then it can be made to vanish, $a = 0$.

5.1.1 Basic cohomologies

In the light of the above discussion, now we pass to the investigation of the solutions to Eqs. (64) and (62)–(63). We have seen that the solution to Eq. (64) belongs to the cohomology of the exterior longitudinal derivative, such that we need to compute $H(\gamma)$ in order to construct the component of highest antighost number from the first-order deformation. This matter is solved with the help of definitions (23)–(29). In order to determine the cohomology $H(\gamma)$, we split the differential γ into two pieces

$$\gamma = \gamma_t + \gamma_r, \quad (65)$$

where γ_t acts non-trivially only on the fields/ghosts from the $(3, 1)$ sector, while γ_r does the same thing, but with respect to the $(2, 2)$ sector. From the above splitting it follows that the nilpotency of γ is equivalent to the nilpotency and anticommutation of its components

$$(\gamma_t)^2 = 0 = (\gamma_r)^2, \quad \gamma_t \gamma_r + \gamma_r \gamma_t = 0, \quad (66)$$

so by Künneth's formula we finally find the isomorphism

$$H(\gamma) = H(\gamma_t) \otimes H(\gamma_r). \quad (67)$$

Using the results from [26] and [27, 28] on the cohomology of the exterior longitudinal derivative, we can state that $H(\gamma)$ is generated on the one hand by $\Theta^{*\Delta}$, $K_{\lambda\mu\nu\xi|\alpha\beta}$ and $F_{\mu\nu\lambda|\alpha\beta\gamma}$ as well as by their spatiotemporal derivatives and, on the other hand, by the ghosts $\mathcal{F}_{\lambda\mu\nu\alpha} \equiv \partial_{[\lambda}\eta_{\mu\nu\alpha]}$, C_ν , $\mathcal{C}_{\mu\nu}$ and $\partial_{[\mu}\mathcal{C}_{\nu\alpha]}$, where $\Theta^{*\Delta}$ denote all the antifields (from both sectors) and

$$K_{\lambda\mu\nu\xi|\alpha\beta} = \partial_\alpha \partial_{[\lambda} t_{\mu\nu\xi]|\beta} - \partial_\beta \partial_{[\lambda} t_{\mu\nu\xi]|\alpha}, \quad (68)$$

$$\begin{aligned} F_{\mu\nu\lambda|\alpha\beta\gamma} = & \partial_\lambda \partial_\gamma r_{\mu\nu|\alpha\beta} + \partial_\mu \partial_\gamma r_{\nu\lambda|\alpha\beta} + \partial_\nu \partial_\gamma r_{\lambda\mu|\alpha\beta} \\ & + \partial_\lambda \partial_\alpha r_{\mu\nu|\beta\gamma} + \partial_\mu \partial_\alpha r_{\nu\lambda|\beta\gamma} + \partial_\nu \partial_\alpha r_{\lambda\mu|\beta\gamma} \\ & + \partial_\lambda \partial_\beta r_{\mu\nu|\gamma\alpha} + \partial_\mu \partial_\beta r_{\nu\lambda|\gamma\alpha} + \partial_\nu \partial_\beta r_{\lambda\mu|\gamma\alpha} \end{aligned} \quad (69)$$

represent the curvature tensors of $t_{\lambda\mu\nu|\alpha}$ and $r_{\mu\nu|\alpha\beta}$. So, the most general, non-trivial representative from $H(\gamma)$ for the overall theory (1) reads as

$$a_I = \alpha_I \left([K_{\lambda\mu\nu\xi|\alpha\beta}], [F_{\mu\nu\lambda|\alpha\beta\gamma}], [\Theta^{*\Delta}] \right) \omega^I (\mathcal{F}_{\lambda\mu\nu\alpha}, \mathcal{C}_{\mu\nu}, \partial_{[\mu}\mathcal{C}_{\nu\alpha]}, C_\nu), \quad (70)$$

where the notation $f([q])$ means that f depends on q and its derivatives up to a finite order, while ω^I denotes the elements of pure ghost number I (and antighost number zero) of a basis in the space of polynomials in the corresponding ghosts and some of their first-order derivatives. The objects α_I (obviously non-trivial in $H^0(\gamma)$) were taken to have a bounded number of derivatives, and therefore they are polynomials in the antifields $\Theta^{*\Delta}$, in the curvature tensors $K_{\lambda\mu\nu\xi|\alpha\beta}$ and $F_{\mu\nu\lambda|\alpha\beta\gamma}$, as well as in their derivatives. Due to their γ -closedness, they are called invariant polynomials. At zero antighost number, the invariant polynomials are polynomials in the curvature tensors $K_{\lambda\mu\nu\xi|\alpha\beta}$ and $F_{\mu\nu\lambda|\alpha\beta\gamma}$ and in their derivatives.

Replacing the solution (70) into Eq. (62) and taking into account definitions (23)–(29), we remark that a necessary (but not sufficient) condition for the existence of (non-trivial) solutions a_{I-1} is that the invariant polynomials α_I are (non-trivial) objects from the local cohomology of the Koszul–Tate differential $H(\delta|d)$ at antighost number $I > 0$ and pure ghost number equal to zero¹, i.e.,

$$\delta\alpha_I = \partial_\mu \binom{(I-1)^\mu}{j}, \quad \text{agh} \left(\binom{(I-1)^\mu}{j} \right) = I - 1 \geq 0, \quad \text{pgh} \left(\binom{(I-1)^\mu}{j} \right) = 0. \quad (71)$$

¹We recall that the local cohomology $H(\delta|d)$ is completely trivial at both strictly positive antighost *and* pure ghost numbers (for instance, see [34], Theorem 5.4 and [35]).

The above notation is generic, in the sense that α_I and $j^{(I-1)\mu}$ may actually carry supplementary Lorentz indices. Consequently, we need to investigate some of the main properties of the local cohomology of the Koszul–Tate differential $H(\delta|d)$ at pure ghost number zero and strictly positive antighost numbers in order to fully determine the component a_I of highest antighost number from the first-order deformation. As the free model under study is a linear gauge theory of Cauchy order equal to four, the general results from [34, 36] (see also [15, 37, 38]) ensure that $H(\delta|d)$ (at pure ghost number zero) is trivial at antighost numbers strictly greater than its Cauchy order

$$H_I(\delta|d) = 0, \quad I > 4. \quad (72)$$

Moreover, if the invariant polynomial α_I , with $\text{agh}(\alpha_I) = I \geq 4$, is trivial in $H_I(\delta|d)$, then it can be taken to be trivial also in $H_I^{\text{inv}}(\delta|d)$

$$\left(\alpha_I = \delta b_{I+1} + \partial_\mu c^{(I)\mu}, \quad \text{agh}(\alpha_I) = I \geq 4 \right) \Rightarrow \alpha_I = \delta \beta_{I+1} + \partial_\mu \gamma^{(I)\mu}, \quad (73)$$

with β_{I+1} and $\gamma^{(I)\mu}$ invariant polynomials. (An element of $H_I^{\text{inv}}(\delta|d)$ is defined via an equation similar to (71), but with the corresponding current also an invariant polynomial.) The result (73) can be proved like in the Appendix B, Theorem 3, from [26]. This is important since together with (72) ensures that the entire local cohomology of the Koszul–Tate differential in the space of invariant polynomials (characteristic cohomology) is trivial in antighost number strictly greater than four

$$H_I^{\text{inv}}(\delta|d) = 0, \quad I > 4. \quad (74)$$

Looking at the definitions (38) involving the transformed antifields (37) and taking into account formulae (33)–(35) with respect to the $(2, 2)$ sector, we can organize the non-trivial representatives of $H_I(\delta|d)$ (at pure ghost number equal to zero) and $H_I^{\text{inv}}(\delta|d)$ with $I \geq 2$ in the following table.

We remark that there is no non-trivial element in $(H_I(\delta|d))_{I \geq 2}$ or $(H_I^{\text{inv}}(\delta|d))_{I \geq 2}$ that effectively involves the curvatures $K_{\lambda\mu\nu\xi|\alpha\beta}$ and $F_{\mu\nu\lambda|\alpha\beta\gamma}$ and/or their derivatives, and the same stands for the quantities that are more than linear in the antifields and/or depend on their derivatives. In contrast to the groups $(H_I(\delta|d))_{I \geq 2}$ and $(H_I^{\text{inv}}(\delta|d))_{I \geq 2}$, which are finite-dimensional, the cohomology $H_1(\delta|d)$ at pure ghost number zero, that is related to global symmetries

Table 1: Non-trivial representatives spanning $H_I(\delta|d)$ and $H_I^{\text{inv}}(\delta|d)$

agh	$H_I(\delta d), H_I^{\text{inv}}(\delta d)$
$I > 4$	none
$I = 4$	$C^{*\mu}$
$I = 3$	$G'^{*\nu\alpha}, \mathcal{C}^{*\mu\nu}$
$I = 2$	$\mathcal{G}'^{*\mu\nu \alpha}, \mathcal{C}^{*\mu\nu \alpha}$

and ordinary conservation laws, is infinite-dimensional since the theory is free.

The previous results on $H(\delta|d)$ and $H^{\text{inv}}(\delta|d)$ at strictly positive antighost numbers are important because they control the obstructions to removing the antifields from the first-order deformation. Indeed, due to (74), it follows that we can successively eliminate all the pieces with $I > 4$ from the non-integrated density of the first-order deformation by adding only trivial terms (the proof is similar to that from the Appendix C in [26]), so we can take, without loss of non-trivial objects, the condition $I \leq 4$ in the decomposition (60). The last representative is of the form (70), where the invariant polynomial is necessarily a non-trivial object from $H_I^{\text{inv}}(\delta|d)$ for $I = 2, 3, 4$ and respectively from $H_1(\delta|d)$ for $I = 1$.

5.1.2 Computation of first-order deformations

Now, we have at hand all the necessary ingredients for computing the general form of the first-order deformation of the solution to the master equation as solution to Eq. (59). In view of this, we decompose the first-order deformation like

$$a = a^{\text{t}} + a^{\text{r}} + a^{\text{t-r}}, \quad (75)$$

where a^{t} denotes the part responsible for the self-interactions of the field $t_{\lambda\mu\nu|\alpha}$, a^{r} is related to the self-interactions of the field $r_{\mu\nu|\alpha\beta}$, and $a^{\text{t-r}}$ signifies the component that describes only the cross-couplings between $t_{\lambda\mu\nu|\alpha}$ and $r_{\mu\nu|\alpha\beta}$. Obviously, Eq. (59) becomes equivalent with three equations, one for each component

$$sa^{\text{t}} = \partial_\mu m_{\text{t}}^\mu, \quad sa^{\text{r}} = \partial_\mu m_{\text{r}}^\mu, \quad sa^{\text{t-r}} = \partial_\mu m_{\text{t-r}}^\mu. \quad (76)$$

The solutions to the first two equations from (76) were investigated in [26] and respectively [27] and read as

$$a^t = 0, \quad a^r = r. \quad (77)$$

In order to solve the third equation from (76), we decompose a^{t-r} along the antighost number like in (60) and stop at $I = 4$

$$a^{t-r} = a_0^{t-r} + a_1^{t-r} + a_2^{t-r} + a_3^{t-r} + a_4^{t-r}, \quad (78)$$

where a_4^{t-r} can be taken as solution to the equation $\gamma a_4^{t-r} = 0$, and therefore it is of the form (70) for $I = 4$, with α_4 an invariant polynomial from $H_4^{\text{inv}}(\delta|d)$. Because $H_4^{\text{inv}}(\delta|d)$ is spanned by $C^{*\mu}$ (see Table 1) and a_4^{t-r} must yield cross-couplings between $t_{\lambda\mu\nu|\alpha}$ and $r_{\mu\nu|\alpha\beta}$ with maximum two spatiotemporal derivatives, it follows that the eligible basis elements at pure ghost number equal to four remain

$$\omega^4 (\mathcal{F}_{\lambda\mu\nu\alpha}, \mathcal{C}_{\mu\nu}, \partial_{[\mu} \mathcal{C}_{\nu\alpha]}, C_\nu) : \mathcal{C}_{\alpha\beta} \mathcal{C}_{\lambda\rho}, \mathcal{C}_{\alpha\beta} \partial_{[\lambda} \mathcal{C}_{\rho\sigma]}. \quad (79)$$

So, up to trivial, γ -exact contributions, we have that

$$a_4^{t-r} = C^{*\mu} (M_\mu^{\alpha\beta\lambda\rho} \mathcal{C}_{\alpha\beta} \mathcal{C}_{\lambda\rho} + N_\mu^{\alpha\beta\lambda\rho\sigma} \mathcal{C}_{\alpha\beta} \partial_{[\lambda} \mathcal{C}_{\rho\sigma]}), \quad (80)$$

where $M_\mu^{\alpha\beta\lambda\rho} = -M_\mu^{\beta\alpha\lambda\rho} = -M_\mu^{\alpha\beta\rho\lambda} = M_\mu^{\lambda\rho\alpha\beta}$ and $N_\mu^{\alpha\beta\lambda\rho\sigma} = N_\mu^{[\alpha\beta]\lambda\rho\sigma} = N_\mu^{\alpha\beta[\lambda\rho\sigma]}$ are some non-derivative, real constants. Replacing a_4^{t-r} into an equation similar to (62) for $I = 4$ and computing δa_4^{t-r} , it follows that

$$\delta a_4^{t-r} = \gamma \lambda_3 + \partial^\mu \tau_\mu - 2G'^{\mu\nu} \partial_{[\nu} \mathcal{C}_{\alpha\beta]} (2M_\mu^{\alpha\beta\lambda\rho} \mathcal{C}_{\lambda\rho} + N_\mu^{\alpha\beta\lambda\rho\sigma} \partial_{[\lambda} \mathcal{C}_{\rho\sigma]}), \quad (81)$$

where

$$\begin{aligned} \lambda_3 = & -G'^{\mu\nu} [2\mathcal{C}_{\alpha\beta|\nu} (2M_\mu^{\alpha\beta\lambda\rho} \mathcal{C}_{\lambda\rho} + N_\mu^{\alpha\beta\lambda\rho\sigma} \partial_{[\lambda} \mathcal{C}_{\rho\sigma]}) \\ & + 3\mathcal{C}_{\alpha\beta} N_\mu^{\alpha\beta\lambda\rho\sigma} \partial_{[\lambda} \mathcal{C}_{\rho\sigma]|\nu}]. \end{aligned} \quad (82)$$

Thus, a_3^{t-r} exists if and only if the third term in the right-hand side of (81) can be written in a γ -exact modulo d form

$$G'^{\mu\nu} \partial_{[\nu} \mathcal{C}_{\alpha\beta]} (2M_\mu^{\alpha\beta\lambda\rho} \mathcal{C}_{\lambda\rho} + N_\mu^{\alpha\beta\lambda\rho\sigma} \partial_{[\lambda} \mathcal{C}_{\rho\sigma]}) = \gamma u_3 + \partial^\mu \pi_\mu. \quad (83)$$

Taking the (left) Euler–Lagrange derivative of the above equation with respect to $G'^{*}\nu\mu$ and recalling the anticommutativity of this operation with γ , we obtain

$$\partial_{[\nu}\mathcal{C}_{\alpha\beta]}(2M_{\mu}^{\alpha\beta\lambda\rho}\mathcal{C}_{\lambda\rho} + N_{\mu}^{\alpha\beta\lambda\rho\sigma}\partial_{[\lambda}\mathcal{C}_{\rho\sigma]}) = \gamma\left(-\frac{\delta^L u_3}{\delta G'^{*}\nu\mu}\right). \quad (84)$$

The last relation shows that the object

$$\partial_{[\nu}\mathcal{C}_{\alpha\beta]}(2M_{\mu}^{\alpha\beta\lambda\rho}\mathcal{C}_{\lambda\rho} + N_{\mu}^{\alpha\beta\lambda\rho\sigma}\partial_{[\lambda}\mathcal{C}_{\rho\sigma]}) , \quad (85)$$

which is a non-trivial element of $H^4(\gamma)$ (see formula (70)), must be γ -exact. This takes place if and only if $M_{\mu}^{\alpha\beta\lambda\rho} = 0 = N_{\mu}^{\alpha\beta\lambda\rho\sigma}$, which further implies

$$a_4^{\text{t-r}} = 0, \quad (86)$$

and hence the first-order deformation in the cross-coupling sector cannot end non-trivially at antighost number $I = 4$.

Consequently, we pass to $I = 3$, in which case we can write

$$a^{\text{t-r}} = a_0^{\text{t-r}} + a_1^{\text{t-r}} + a_2^{\text{t-r}} + a_3^{\text{t-r}}. \quad (87)$$

Here, $a_3^{\text{t-r}}$ is solution to the equation $\gamma a_3^{\text{t-r}} = 0$, and thus is of the type (70) for $I = 3$, with α_3 an invariant polynomial from $H_3^{\text{inv}}(\delta|d)$. There are three independent candidates that comply with all the hypotheses (including that on the derivative order of the interacting Lagrangian)

$$a_3^{\text{t-r}} = G'^{*}\nu\alpha M_{\nu\alpha}^{\lambda\rho\sigma\tau\gamma\delta}\mathcal{C}_{\lambda\rho}\mathcal{F}_{\sigma\tau\gamma\delta} + \mathcal{C}^{*\tau\nu}(N_{\tau\nu}^{\alpha}C_{\alpha} + L_{\tau\nu}^{\lambda\rho\sigma\xi\gamma\delta}\mathcal{C}_{\lambda\rho}\mathcal{F}_{\sigma\xi\gamma\delta}), \quad (88)$$

where $M_{\nu\alpha}^{\lambda\rho\sigma\tau\gamma\delta}$, $N_{\tau\nu}^{\alpha}$ and $L_{\tau\nu}^{\lambda\rho\sigma\xi\gamma\delta}$ are some non-derivative, real constants. By direct computation it follows that

$$\begin{aligned} \delta a_3^{\text{t-r}} &= \gamma\lambda_2 + \partial^{\mu}\sigma_{\mu} \\ &+ \left(\frac{2}{3}\mathcal{G}'^{*\mu\nu|\alpha}M_{\nu\alpha}^{\lambda\rho\sigma\xi\gamma\delta} + \mathcal{C}^{*\tau\nu|\mu}L_{\tau\nu}^{\lambda\rho\sigma\xi\gamma\delta}\right)\partial_{[\mu}\mathcal{C}_{\lambda\rho]}\mathcal{F}_{\sigma\xi\gamma\delta}, \end{aligned} \quad (89)$$

with

$$\begin{aligned} \lambda_2 &= \left(\frac{2}{3}\mathcal{G}'^{*\mu\nu|\alpha}M_{\nu\alpha}^{\lambda\rho\sigma\tau\gamma\delta} + \mathcal{C}^{*\xi\nu|\mu}L_{\xi\nu}^{\lambda\rho\sigma\tau\gamma\delta}\right)(-\mathcal{C}_{\lambda\rho|\mu}\mathcal{F}_{\sigma\tau\gamma\delta} \\ &+ \partial_{[\sigma}t_{\tau\gamma\delta]|\mu}\mathcal{C}_{\lambda\rho}) - \frac{1}{2}\mathcal{C}^{*\tau\nu|\mu}N_{\tau\nu}^{\alpha}G'_{\mu\alpha}. \end{aligned} \quad (90)$$

From (89) we find that $a_2^{\text{t-r}}$ exists if and only if the third term in the right-hand side of (89) can be written in a γ -exact modulo d form

$$\left(\frac{2}{3} \mathcal{G}'^{*\mu\nu|\alpha} M_{\nu\alpha}^{\lambda\rho\sigma\tau\gamma\delta} + \mathcal{C}^{*\xi\nu|\mu} L_{\xi\nu}^{\lambda\rho\sigma\tau\gamma\delta} \right) \partial_{[\mu} \mathcal{C}_{\lambda\rho]} \mathcal{F}_{\sigma\tau\gamma\delta} = \gamma u_2 + \partial^\mu l_\mu. \quad (91)$$

Taking successively the Euler–Lagrange derivatives of (91) with respect to the bosonic antifields $\mathcal{G}'^{*\mu\nu|\alpha}$ and $\mathcal{C}^{*\xi\nu|\mu}$ and taking into account the commutativity of this operations with γ , we find

$$\frac{2}{3} M_{\nu\alpha}^{\lambda\rho\sigma\tau\gamma\delta} \partial_{[\mu} \mathcal{C}_{\lambda\rho]} \mathcal{F}_{\sigma\tau\gamma\delta} = \gamma \left(\frac{\delta^L u_2}{\delta \mathcal{G}'^{*\mu\nu|\alpha}} \right), \quad L_{\xi\nu}^{\lambda\rho\sigma\tau\gamma\delta} \partial_{[\mu} \mathcal{C}_{\lambda\rho]} \mathcal{F}_{\sigma\tau\gamma\delta} = \gamma \left(\frac{\delta^L u_2}{\delta \mathcal{C}^{*\xi\nu|\mu}} \right). \quad (92)$$

The previous equations indicate that the objects

$$\frac{2}{3} M_{\nu\alpha}^{\lambda\rho\sigma\tau\gamma\delta} \partial_{[\mu} \mathcal{C}_{\lambda\rho]} \mathcal{F}_{\sigma\tau\gamma\delta}, \quad L_{\xi\nu}^{\lambda\rho\sigma\tau\gamma\delta} \partial_{[\mu} \mathcal{C}_{\lambda\rho]} \mathcal{F}_{\sigma\tau\gamma\delta}, \quad (93)$$

which are non-trivial elements of $H^3(\gamma)$ (see relation (70)), must be γ -exact. This takes place if and only if $M_{\nu\alpha}^{\lambda\rho\sigma\tau\gamma\delta} = 0 = L_{\xi\nu}^{\lambda\rho\sigma\tau\gamma\delta}$, which further leads to

$$a_3^{\text{t-r}} = \mathcal{C}^{*\tau\nu} N_{\tau\nu}^\alpha C_\alpha. \quad (94)$$

Then, Eq. (89) takes the form $\delta a_3^{\text{t-r}} = \gamma \left(-\frac{1}{2} \mathcal{C}^{*\tau\nu|\mu} N_{\tau\nu}^\alpha G'_{\mu\alpha} \right) + \partial^\mu \sigma_\mu$, and hence

$$a_2^{\text{t-r}} = \frac{1}{2} \mathcal{C}^{*\tau\nu|\mu} N_{\tau\nu}^\alpha G'_{\mu\alpha} + \bar{a}_2^{\text{t-r}}. \quad (95)$$

In the above $\bar{a}_2^{\text{t-r}}$ stands for the general solution to the homogeneous equation $\gamma \bar{a}_2^{\text{t-r}} = 0$. It is easy to see that the only covariant choice of the non-derivative, real constants $N_{\tau\nu}^\alpha$ is given by $N_{\tau\nu}^\alpha = c \varepsilon^{\alpha}{}_{\tau\nu} = c \sigma^{\alpha\beta} \varepsilon_{\beta\tau\nu}$, where $\varepsilon_{\beta\tau\nu}$ represents the Levi–Civita symbol in $D = 3$ and c is a real constant. Because we work in $D \geq 5$, it follows that $c = 0$, so $N_{\tau\nu}^\alpha = 0$. Inserting this result in formulae (94)–(95), we then get

$$a_3^{\text{t-r}} = 0, \quad a_2^{\text{t-r}} = \bar{a}_2^{\text{t-r}}, \quad (96)$$

$$\gamma \bar{a}_2^{\text{t-r}} = 0 = \gamma a_2^{\text{t-r}}. \quad (97)$$

In consequence, the first-order deformation cannot end non-trivially at antighost number three either.

As a consequence, we can write

$$a^{\text{t-r}} = a_0^{\text{t-r}} + a_1^{\text{t-r}} + a_2^{\text{t-r}}, \quad (98)$$

with $a_2^{\text{t-r}}$ the general solution to the homogeneous equation $\gamma a_2^{\text{t-r}} = 0$, and thus of the type (70) for $I = 2$, with α_2 an invariant polynomial from $H_2^{\text{inv}}(\delta|d)$. There appear two distinct solutions that fulfill all the working hypotheses, namely

$$a_2^{\text{t-r}} = \mathcal{G}^{I*\mu\nu|\alpha} \left(P_{\mu\nu\alpha}^{\lambda\rho} \mathcal{C}_{\lambda\rho} + Q_{\mu\nu\alpha}^{\lambda\rho\sigma} \partial_{[\lambda} \mathcal{C}_{\rho\sigma]} \right), \quad (99)$$

where $P_{\mu\nu\alpha}^{\lambda\rho}$ and $Q_{\mu\nu\alpha}^{\lambda\rho\sigma}$ are some non-derivative, real constants, with the properties $P_{\mu\nu\alpha}^{\lambda\rho} = -P_{\mu\nu\alpha}^{\rho\lambda}$ and $Q_{\mu\nu\alpha}^{\lambda\rho\sigma} = Q_{\mu\nu\alpha}^{[\lambda\rho\sigma]}$. Acting with δ on (99), we infer

$$\delta a_2^{\text{t-r}} = \gamma \lambda_1 + \partial^\mu k_\mu + t^{*\tau\mu\nu|\alpha} P_{\mu\nu\alpha}^{\lambda\rho} \partial_{[\tau} \mathcal{C}_{\lambda\rho]}, \quad (100)$$

where

$$\lambda_1 = t^{*\tau\mu\nu|\alpha} P_{\mu\nu\alpha}^{\lambda\rho} \mathcal{C}_{\lambda\rho|\tau} + \frac{3}{2} t^{*\tau\mu\nu|\alpha} Q_{\mu\nu\alpha}^{\lambda\rho\sigma} \partial_{[\lambda} \mathcal{C}_{\rho\sigma]|\tau}. \quad (101)$$

From (100) we find that $a_1^{\text{t-r}}$ exists if and only if the third term in the right-hand side of (100) can be written in a γ -exact modulo d form

$$t^{*\tau\mu\nu|\alpha} P_{\mu\nu\alpha}^{\lambda\rho} \partial_{[\tau} \mathcal{C}_{\lambda\rho]} = \gamma u_1 + \partial^\mu q_\mu. \quad (102)$$

Taking the (left) Euler–Lagrange derivative of the above equation with respect to $t^{*\tau\mu\nu|\alpha}$ and recalling the anticommutativity of this operation with γ , we deduce

$$P_{\mu\nu\alpha}^{\lambda\rho} \partial_{[\tau} \mathcal{C}_{\lambda\rho]} = \gamma \left(-\frac{\delta^L u_1}{\delta t^{*\tau\mu\nu|\alpha}} \right). \quad (103)$$

The previous equation reduces to the requirement that the object

$$P_{\mu\nu\alpha}^{\lambda\rho} \partial_{[\tau} \mathcal{C}_{\lambda\rho]}, \quad (104)$$

which is a non-trivial element of $H^2(\gamma)$ (see relation (70)), must be γ -exact. This holds if and only if $P_{\mu\nu\alpha}^{\lambda\rho} = 0$. The last result replaced in formulae (99)–(101) yields

$$a_2^{\text{t-r}} = \mathcal{G}^{I*\mu\nu|\alpha} Q_{\mu\nu\alpha}^{\lambda\rho\sigma} \partial_{[\lambda} \mathcal{C}_{\rho\sigma]}, \quad (105)$$

$$\delta a_2^{\text{t-r}} = \gamma \left(\frac{3}{2} t^{*\tau\mu\nu|\alpha} Q_{\mu\nu\alpha}^{\lambda\rho\sigma} \partial_{[\lambda} \mathcal{C}_{\rho\sigma]|\tau} \right) + \partial^\mu k_\mu. \quad (106)$$

Next, Eq. (106) produces in a simple manner the corresponding $a_1^{\text{t-r}}$

$$a_1^{\text{t-r}} = -\frac{3}{2}t^{*\tau\mu\nu|\alpha}Q_{\mu\nu\alpha}^{\lambda\rho\sigma}\partial_{[\lambda}\mathcal{C}_{\rho\sigma]|\tau} + \bar{a}_1^{\text{t-r}}, \quad (107)$$

where $\bar{a}_1^{\text{t-r}}$ means the general solution to the homogeneous equation $\gamma\bar{a}_1^{\text{t-r}} = 0$. Recalling the working hypotheses, we conclude that

$$\bar{a}_1^{\text{t-r}} = r^{*\mu\nu|\alpha\beta}Z_{\mu\nu\alpha\beta}^{\sigma\tau\gamma\delta}\mathcal{F}_{\sigma\tau\gamma\delta}, \quad (108)$$

where $Z_{\mu\nu\alpha\beta}^{\sigma\tau\gamma\delta}$ denote some real, non-derivative constants, which are completely antisymmetric with respect to the indices $\{\sigma, \tau, \gamma, \delta\}$. Due to the mixed symmetry properties of $t^{*\mu\nu|\alpha\tau}$ and $r^{*\mu\nu|\alpha\beta}$, the only covariant choice of $Q_{\mu\nu\alpha}^{\lambda\rho\sigma}$ and $Z_{\mu\nu\alpha\beta}^{\sigma\tau\gamma\delta}$ in $D \geq 5$ that does not end up with trivial solutions reads as

$$Q_{\mu\nu\alpha}^{\lambda\rho\sigma} = \frac{4}{3}\varepsilon_{\mu\nu\alpha}{}^{\lambda\rho\sigma} = \frac{4}{3}\sigma^{\lambda\lambda'}\sigma^{\rho\rho'}\sigma^{\sigma\sigma'}\varepsilon_{\mu\nu\alpha\lambda'\rho'\sigma'}, \quad Z_{\mu\nu\alpha\beta}^{\sigma\tau\gamma\delta} = 0, \quad (109)$$

with $\varepsilon_{\mu\nu\alpha\lambda'\rho'\sigma'}$ the six-dimensional Levi-Civita symbol. Inserting (109) into (105) and (107)–(108) and recalling transformations (37), we finally obtain

$$\begin{aligned} a_2^{\text{t-r}} &= \varepsilon^{\lambda\mu\nu\alpha\beta\gamma}\eta_{\lambda\mu\nu}^*\partial_\alpha\mathcal{C}_{\beta\gamma}, \\ a_1^{\text{t-r}} &= -2\varepsilon_{\lambda\mu\nu\rho\beta\gamma}t^{*\lambda\mu\nu|\alpha}\left(\partial^\rho\mathcal{C}^{\beta\gamma|}{}_\alpha - \frac{1}{4}\delta^\gamma_\alpha\partial^{[\rho}\mathcal{C}^{\beta\tau]|\tau}{}_\tau\right), \quad \bar{a}_1^{\text{t-r}} = 0 \end{aligned} \quad (110)$$

The second term from the right-hand side of $a_1^{\text{t-r}}$ is vanishing. Nevertheless, it has been introduced in order to restore the mixed symmetry (3, 1) of the quantity $\delta a_1^{\text{t-r}}/\delta t^{*\lambda\mu\nu|\alpha}$. By means of (111), we infer

$$\delta a_1^{\text{t-r}} = \gamma\left[2\varepsilon^{\lambda\mu\nu\alpha\beta\gamma}t_{\lambda\mu\nu|\rho}\left(\partial_\sigma\partial_\alpha r_{\beta\gamma|}{}^{\sigma\rho} - \frac{1}{2}\delta^\rho_\gamma\partial^\tau\partial_\alpha r_{\beta\tau}\right)\right] + \partial^\mu p_\mu, \quad (112)$$

which then yields

$$a_0^{\text{t-r}} = -2\varepsilon^{\lambda\mu\nu\alpha\beta\gamma}t_{\lambda\mu\nu|\rho}\left(\partial_\sigma\partial_\alpha r_{\beta\gamma|}{}^{\sigma\rho} - \frac{1}{2}\delta^\rho_\gamma\partial^\tau\partial_\alpha r_{\beta\tau}\right) + \bar{a}_0^{\text{t-r}}, \quad (113)$$

where $\bar{a}_0^{\text{t-r}}$ is the general solution to the ‘homogeneous’ equation

$$\gamma\bar{a}_0^{\text{t-r}} = \partial_\mu m^\mu. \quad (114)$$

Next, we investigate the solutions to (114). There are two main types of solutions to this equation. The first type, to be denoted by \bar{a}_0^{t-r} , corresponds to $m^\mu = 0$ and is given by gauge-invariant, non-integrated densities constructed out of the original fields and their spatiotemporal derivatives, which, according to (70), are of the form $\bar{a}_0^{t-r} = \bar{a}_0^{t-r} ([K_{\lambda\mu\nu\xi|\alpha\beta}], [F_{\mu\nu\lambda|\alpha\beta\gamma}])$, up to the condition that they effectively describe cross-couplings between the two types of fields and cannot be written in a divergence-like form. Such a solution implies at least four derivatives of the fields and consequently must be forbidden by setting $\bar{a}_0^{t-r} = 0$.

The second kind of solutions is associated with $m^\mu \neq 0$ in (114), being understood that we discard the divergence-like quantities and maintain the condition on the maximum derivative order of the interacting Lagrangian being equal to two. In order to solve this equation we start from the requirement that \bar{a}_0^{t-r} may contain at most two derivatives, so it can be decomposed like

$$\bar{a}_0^{t-r} = \omega_0 + \omega_1 + \omega_2, \quad (115)$$

where $(\omega_i)_{i=\overline{0,2}}$ contains i derivatives. Due to the different number of derivatives in the components ω_0 , ω_1 , and ω_2 , Eq. (115) is equivalent to three independent equations

$$\gamma\omega_k = \partial_\mu j_k^\mu, \quad k = 0, 1, 2. \quad (116)$$

Eq. (116) for $k = 0$ implies the (necessary) conditions

$$\partial_\lambda \left(\frac{\partial\omega_0}{\partial t_{\lambda\mu\nu|\alpha}} \right) = 0, \quad \partial_\alpha \left(\frac{\partial\omega_0}{\partial t_{\lambda\mu\nu|\alpha}} \right) = 0, \quad \partial_\mu \left(\frac{\partial\omega_0}{\partial r_{\mu\nu|\alpha\beta}} \right) = 0. \quad (117)$$

The last equation from (117) possesses only the constant solution

$$\frac{\partial\omega_0}{\partial r_{\mu\nu|\alpha\beta}} = k (\sigma^{\mu\alpha}\sigma^{\nu\beta} - \sigma^{\mu\beta}\sigma^{\nu\alpha}), \quad (118)$$

where k is a real constant, so we find that

$$\omega_0 = 2kr + B (t_{\lambda\mu\nu|\alpha}). \quad (119)$$

Since ω_0 provides no cross-couplings between $t_{\lambda\mu\nu|\alpha}$ and $r_{\mu\nu|\alpha\beta}$, we can take

$$\omega_0 = 0 \quad (120)$$

in (115).

As a digression, we note that the general solution to the equations

$$\partial_\lambda \bar{T}^{\lambda\mu\nu|\alpha} = 0, \quad \partial_\alpha \bar{T}^{\lambda\mu\nu|\alpha} = 0 \quad (121)$$

(with $\bar{T}^{\lambda\mu\nu|\alpha}$ a covariant tensor field with the mixed symmetry $(3, 1)$) reads as [26]

$$\bar{T}^{\lambda\mu\nu|\alpha} = \partial_\xi \partial_\beta \bar{\Phi}^{\lambda\mu\nu\xi|\alpha\beta}, \quad (122)$$

where $\bar{\Phi}^{\rho\lambda\mu\nu|\beta\alpha}$ is a tensor with the mixed symmetry $(4, 2)$. A constant solution $C^{\lambda\mu\nu|\alpha}$ is excluded from covariance arguments due to the mixed symmetry $(3, 1)$. Along the same line, the general solution to the equations

$$\partial_\mu \bar{R}^{\mu\nu|\alpha\beta} = 0 \quad (123)$$

(with $\bar{R}^{\mu\nu|\alpha\beta}$ a covariant tensor field with the mixed symmetry $(2, 2)$) is represented by [27]

$$\bar{R}^{\mu\nu|\alpha\beta} = \partial_\rho \partial_\gamma \bar{\Omega}^{\mu\nu\rho|\alpha\beta\gamma} + k (\sigma^{\mu\alpha} \sigma^{\nu\beta} - \sigma^{\mu\beta} \sigma^{\nu\alpha}), \quad (124)$$

where $\bar{\Omega}^{\mu\nu\rho|\alpha\beta\gamma}$ is a tensor with the mixed symmetry $(3, 3)$ and k is an arbitrary, real constant. Now, it is clear why the solution to the last equation from (117) reduces to (118): $\partial\omega_0/\partial r_{\mu\nu|\alpha\beta}$ displays the mixed symmetry $(2, 2)$, but is derivative-free by assumption, so a term similar to the first one from the right-hand side of (124) is forbidden.

Eq. (116) for $k = 1$ leads to the requirements

$$\partial_\lambda \left(\frac{\delta\omega_1}{\delta t_{\lambda\mu\nu|\alpha}} \right) = 0, \quad \partial_\alpha \left(\frac{\delta\omega_1}{\delta t_{\lambda\mu\nu|\alpha}} \right) = 0, \quad \partial_\mu \left(\frac{\delta\omega_1}{\delta r_{\mu\nu|\alpha\beta}} \right) = 0, \quad (125)$$

where $\delta\omega_1/\delta t_{\lambda\mu\nu|\alpha}$ and $\delta\omega_1/\delta r_{\mu\nu|\alpha\beta}$ denote the Euler–Lagrange derivatives of ω_1 with respect to the corresponding fields. Looking at (122) and (124) and recalling that ω_1 is by hypothesis of order one in the spatiotemporal derivatives of the fields, the only solution to equations (125) reduces to

$$\frac{\delta\omega_1}{\delta r_{\mu\nu|\alpha\beta}} = 0 = \frac{\delta\omega_1}{\delta t_{\lambda\mu\nu|\alpha}}. \quad (126)$$

This solution forbids the cross-couplings between the two types of fields, so we can safely take

$$\omega_1 = 0. \quad (127)$$

Finally, we pass to Eq. (116) for $k = 2$, which produces the restrictions

$$\partial_\lambda \left(\frac{\delta\omega_2}{\delta t_{\lambda\mu\nu|\alpha}} \right) = 0, \quad \partial_\alpha \left(\frac{\delta\omega_2}{\delta t_{\lambda\mu\nu|\alpha}} \right) = 0, \quad \partial_\mu \left(\frac{\delta\omega_2}{\delta r_{\mu\nu|\alpha\beta}} \right) = 0, \quad (128)$$

with the solutions (see formulae (122) and (124))

$$\frac{\delta\omega_2}{\delta t_{\lambda\mu\nu|\alpha}} = \partial_\gamma \partial_\sigma W^{\lambda\mu\nu\gamma|\alpha\sigma}, \quad \frac{\delta\omega_2}{\delta r_{\mu\nu|\alpha\beta}} = \partial_\gamma \partial_\sigma U^{\mu\nu\gamma|\alpha\beta\sigma}. \quad (129)$$

The tensor $W^{\lambda\mu\nu\gamma|\alpha\sigma}$ has the mixed symmetry of the curvature tensor $K^{\lambda\mu\nu\gamma|\alpha\sigma}$ and the tensor $U^{\mu\nu\gamma|\alpha\beta\sigma}$ exhibits the mixed symmetry of the curvature tensor $F^{\mu\nu\gamma|\alpha\beta\sigma}$. Both of them are derivative-free since ω_2 contains precisely two derivatives of the fields. At this stage it is useful to introduce a derivation in the algebra of the fields and of their derivatives that counts the powers of the fields and of their derivatives

$$N = \sum_{k \geq 0} \left[(\partial_{\mu_1} \cdots \partial_{\mu_k} t_{\lambda\mu\nu|\alpha}) \frac{\partial}{\partial (\partial_{\mu_1} \cdots \partial_{\mu_k} t_{\lambda\mu\nu|\alpha})} + (\partial_{\mu_1} \cdots \partial_{\mu_k} r_{\mu\nu|\alpha\beta}) \frac{\partial}{\partial (\partial_{\mu_1} \cdots \partial_{\mu_k} r_{\mu\nu|\alpha\beta})} \right], \quad (130)$$

so for every non-integrated density ρ we have that

$$N\rho = t_{\lambda\mu\nu|\alpha} \frac{\delta\rho}{\delta t_{\lambda\mu\nu|\alpha}} + r_{\mu\nu|\alpha\beta} \frac{\delta\rho}{\delta r_{\mu\nu|\alpha\beta}} + \partial_\mu s^\mu, \quad (131)$$

where $\delta\rho/\delta t_{\mu\nu|\alpha\beta}$ and $\delta\rho/\delta r_{\mu\nu|\alpha\beta}$ denote the variational derivatives of ρ with respect to the fields. If $\rho^{(l)}$ is a homogeneous polynomial of order $l > 0$ in the fields $\{t_{\lambda\mu\nu|\alpha}, r_{\mu\nu|\alpha\beta}\}$ and their derivatives, then $N\rho^{(l)} = l\rho^{(l)}$. Using (129) and (131), we find that

$$N\omega_2 = \frac{1}{8} K_{\lambda\mu\nu\gamma|\alpha\sigma} W^{\lambda\mu\nu\gamma|\alpha\sigma} + \frac{1}{9} F_{\mu\nu\gamma|\alpha\beta\sigma} U^{\mu\nu\gamma|\alpha\beta\sigma} + \partial_\mu v^\mu. \quad (132)$$

We expand ω_2 according to the various eigenvalues of N like

$$\omega_2 = \sum_{l > 0} \omega_2^{(l)}, \quad (133)$$

where $N\omega_2^{(l)} = l\omega_2^{(l)}$, such that

$$N\omega_2 = \sum_{l>0} l\omega_2^{(l)}. \quad (134)$$

Comparing (132) with (134), we reach the conclusion that the decomposition (133) induces a similar decomposition with respect to $W^{\lambda\mu\nu\gamma|\alpha\sigma}$ and $U^{\mu\nu\gamma|\alpha\beta\sigma}$

$$W^{\lambda\mu\nu\gamma|\alpha\sigma} = \sum_{l>0} W_{(l-1)}^{\lambda\mu\nu\gamma|\alpha\sigma}, \quad U^{\mu\nu\gamma|\alpha\beta\sigma} = \sum_{l>0} U_{(l-1)}^{\mu\nu\gamma|\alpha\beta\sigma}. \quad (135)$$

Substituting (135) into (132) and comparing the resulting expression with (134), we obtain that

$$\omega_2^{(l)} = \frac{1}{8l} K_{\lambda\mu\nu\gamma|\alpha\sigma} W_{(l-1)}^{\lambda\mu\nu\gamma|\alpha\sigma} + \frac{1}{9l} F_{\mu\nu\gamma|\alpha\beta\sigma} U_{(l-1)}^{\mu\nu\gamma|\alpha\beta\sigma} + \partial_\mu \bar{v}_{(l)}^\mu. \quad (136)$$

Introducing (136) in (133), we arrive at

$$\omega_2 = K_{\lambda\mu\nu\gamma|\alpha\sigma} \bar{W}^{\lambda\mu\nu\gamma|\alpha\sigma} + F_{\mu\nu\gamma|\alpha\beta\sigma} \bar{U}^{\mu\nu\gamma|\alpha\beta\sigma} + \partial_\mu \bar{v}^\mu, \quad (137)$$

where

$$\bar{W}^{\lambda\mu\nu\gamma|\alpha\sigma} = \sum_{l>0} \frac{1}{8l} W_{(l-1)}^{\lambda\mu\nu\gamma|\alpha\sigma}, \quad \bar{U}^{\mu\nu\gamma|\alpha\beta\sigma} = \sum_{l>0} \frac{1}{9l} U_{(l-1)}^{\mu\nu\gamma|\alpha\beta\sigma}. \quad (138)$$

Applying γ on (137), we infer that a necessary condition for the existence of solutions to the equation $\gamma\omega_2 = \partial_\mu j_2^\mu$ is that the functions $\bar{W}^{\lambda\mu\nu\gamma|\alpha\sigma}$ and $\bar{U}^{\mu\nu\gamma|\alpha\beta\sigma}$ entering (137) must satisfy the equations

$$\partial_\rho \left(F_{\mu\nu\gamma|\alpha\beta\sigma} \frac{\partial \bar{U}^{\mu\nu\gamma|\alpha\beta\sigma}}{\partial r_{\rho\delta|\xi\chi}} + K_{\lambda\mu\nu\gamma|\alpha\sigma} \frac{\partial \bar{W}^{\lambda\mu\nu\gamma|\alpha\sigma}}{\partial r_{\rho\delta|\xi\chi}} \right) = 0, \quad (139)$$

$$\partial_\chi \left(F_{\mu\nu\gamma|\alpha\beta\sigma} \frac{\partial \bar{U}^{\mu\nu\gamma|\alpha\beta\sigma}}{\partial t_{\rho\delta\xi|\chi}} + K_{\lambda\mu\nu\gamma|\alpha\sigma} \frac{\partial \bar{W}^{\lambda\mu\nu\gamma|\alpha\sigma}}{\partial t_{\rho\delta\xi|\chi}} \right) = 0, \quad (140)$$

$$\partial_\rho \left(F_{\mu\nu\gamma|\alpha\beta\sigma} \frac{\partial \bar{U}^{\mu\nu\gamma|\alpha\beta\sigma}}{\partial t_{\rho\delta\xi|\chi}} + K_{\lambda\mu\nu\gamma|\alpha\sigma} \frac{\partial \bar{W}^{\lambda\mu\nu\gamma|\alpha\sigma}}{\partial t_{\rho\delta\xi|\chi}} \right) = 0. \quad (141)$$

The general solution to Eqs. (139)–(141) reads as

$$F_{\mu\nu\gamma|\alpha\beta\sigma} \frac{\partial \bar{U}^{\mu\nu\gamma|\alpha\beta\sigma}}{\partial r_{\rho\delta|\xi\chi}} + K_{\lambda\mu\nu\gamma|\alpha\sigma} \frac{\partial \bar{W}^{\lambda\mu\nu\gamma|\alpha\sigma}}{\partial r_{\rho\delta|\xi\chi}} = \partial_\tau \partial_\theta E^{\rho\delta\tau|\xi\chi\theta}, \quad (142)$$

$$F_{\mu\nu\gamma|\alpha\beta\sigma} \frac{\partial \bar{U}^{\mu\nu\gamma|\alpha\beta\sigma}}{\partial t_{\rho\delta\xi|\chi}} + K_{\lambda\mu\nu\gamma|\alpha\sigma} \frac{\partial \bar{W}^{\lambda\mu\nu\gamma|\alpha\sigma}}{\partial t_{\rho\delta\xi|\chi}} = \partial_\tau \partial_\theta H^{\rho\delta\xi\tau|\chi\theta}, \quad (143)$$

where the functions $E^{\rho\delta\tau|\xi\chi\theta}$ and $H^{\rho\delta\xi\tau|\chi\theta}$ are derivative-free and exhibit the mixed symmetries (3,3) and (4,2) respectively. By direct computations we deduce

$$\begin{aligned} \partial_\tau \partial_\theta E^{\rho\delta\tau|\xi\chi\theta} &= \frac{\partial^2 E^{\rho\delta\tau|\xi\chi\theta}}{\partial r_{\rho'\delta'|\xi'\chi'} \partial r_{\rho''\delta''|\xi''\chi''}} (\partial_\theta r_{\rho'\delta'|\xi'\chi'}) (\partial_\tau r_{\rho''\delta''|\xi''\chi''}) \\ &+ \frac{\partial^2 E^{\rho\delta\tau|\xi\chi\theta}}{\partial t_{\rho'\delta'\xi'|\chi'} \partial t_{\rho''\delta''\xi''|\chi''}} (\partial_\theta t_{\rho'\delta'\xi'|\chi'}) (\partial_\tau t_{\rho''\delta''\xi''|\chi''}) \\ &+ \frac{\partial^2 (E^{\rho\delta\tau|\xi\chi\theta} + E^{\rho\delta\theta|\xi\chi\tau})}{\partial r_{\rho'\delta'|\xi'\chi'} \partial t_{\rho''\delta''\xi''|\chi''}} (\partial_\theta r_{\rho'\delta'|\xi'\chi'}) (\partial_\tau t_{\rho''\delta''\xi''|\chi''}) \\ &+ \frac{\partial E^{\rho\delta\tau|\xi\chi\theta}}{\partial r_{\rho'\delta'|\xi'\chi'}} \partial_\tau \partial_\theta r_{\rho'\delta'|\xi'\chi'} + \frac{\partial E^{\rho\delta\tau|\xi\chi\theta}}{\partial t_{\rho'\delta'\xi'|\chi'}} \partial_\tau \partial_\theta t_{\rho'\delta'\xi'|\chi'}, \quad (144) \end{aligned}$$

$$\begin{aligned} \partial_\tau \partial_\theta H^{\rho\delta\xi\tau|\chi\theta} &= \frac{\partial^2 H^{\rho\delta\xi\tau|\chi\theta}}{\partial r_{\rho'\delta'|\xi'\chi'} \partial r_{\rho''\delta''|\xi''\chi''}} (\partial_\theta r_{\rho'\delta'|\xi'\chi'}) (\partial_\tau r_{\rho''\delta''|\xi''\chi''}) \\ &+ \frac{\partial^2 H^{\rho\delta\xi\tau|\chi\theta}}{\partial t_{\rho'\delta'\xi'|\chi'} \partial t_{\rho''\delta''\xi''|\chi''}} (\partial_\theta t_{\rho'\delta'\xi'|\chi'}) (\partial_\tau t_{\rho''\delta''\xi''|\chi''}) \\ &+ \frac{\partial^2 (H^{\rho\delta\xi\tau|\chi\theta} + H^{\rho\delta\xi\theta|\chi\tau})}{\partial r_{\rho'\delta'|\xi'\chi'} \partial t_{\rho''\delta''\xi''|\chi''}} (\partial_\theta r_{\rho'\delta'|\xi'\chi'}) (\partial_\tau t_{\rho''\delta''\xi''|\chi''}) \\ &+ \frac{\partial H^{\rho\delta\xi\tau|\chi\theta}}{\partial r_{\rho'\delta'|\xi'\chi'}} \partial_\tau \partial_\theta r_{\rho'\delta'|\xi'\chi'} + \frac{\partial H^{\rho\delta\xi\tau|\chi\theta}}{\partial t_{\rho'\delta'\xi'|\chi'}} \partial_\tau \partial_\theta t_{\rho'\delta'\xi'|\chi'}. \quad (145) \end{aligned}$$

Substituting (144)–(145) in (142)–(143) and comparing the left-hand sides with the corresponding right-hand sides of the resulting relations, we find the necessary equations

$$\frac{\partial^2 E^{\rho\delta\tau|\xi\chi\theta}}{\partial r_{\rho'\delta'|\xi'\chi'} \partial r_{\rho''\delta''|\xi''\chi''}} = 0, \quad \frac{\partial^2 E^{\rho\delta\tau|\xi\chi\theta}}{\partial t_{\rho'\delta'\xi'|\chi'} \partial t_{\rho''\delta''\xi''|\chi''}} = 0, \quad (146)$$

$$\frac{\partial^2 H^{\rho\delta\xi\tau|\chi\theta}}{\partial r_{\rho'\delta'|\xi'\chi'} \partial r_{\rho''\delta''|\xi''\chi''}} = 0, \quad \frac{\partial^2 H^{\rho\delta\xi\tau|\chi\theta}}{\partial t_{\rho'\delta'\xi'|\chi'} \partial t_{\rho''\delta''\xi''|\chi''}} = 0, \quad (147)$$

$$\frac{\partial^2 (E^{\rho\delta\tau|\xi\chi\theta} + E^{\rho\delta\theta|\xi\chi\tau})}{\partial r_{\rho'\delta'|\xi'\chi'} \partial t_{\rho''\delta''\xi''|\chi''}} = 0, \quad \frac{\partial^2 (H^{\rho\delta\xi\tau|\chi\theta} + H^{\rho\delta\xi\theta|\chi\tau})}{\partial r_{\rho'\delta'|\xi'\chi'} \partial t_{\rho''\delta''\xi''|\chi''}} = 0. \quad (148)$$

The above relations allow us to write

$$\frac{1}{2} (E^{\rho\delta\tau|\xi\chi\theta} + E^{\rho\delta\theta|\xi\chi\tau}) = C^{\rho\delta\tau|\xi\chi\theta;\rho'\delta'|\xi'\chi'} r_{\rho'\delta'|\xi'\chi'} + C^{\rho\delta\theta|\xi\chi\tau;\rho'\delta'|\xi'\chi'} t_{\rho'\delta'|\xi'\chi'}, \quad (149)$$

$$\frac{1}{2} (H^{\rho\delta\xi\tau|\chi\theta} + H^{\rho\delta\xi\theta|\chi\tau}) = \hat{C}^{\rho\delta\xi\tau|\chi\theta;\rho'\delta'|\xi'\chi'} r_{\rho'\delta'|\xi'\chi'} + \hat{C}^{\rho\delta\xi\tau|\chi\theta;\rho'\delta'|\xi'\chi'} t_{\rho'\delta'|\xi'\chi'}, \quad (150)$$

where the quantities denoted by C or \hat{C} are some non-derivative, real tensors, with the expressions

$$C^{\rho\delta\tau|\xi\chi\theta;\rho'\delta'|\xi'\chi'} = \tilde{C}^{\rho\delta\tau|\xi\chi\theta;\rho'\delta'|\xi'\chi'} + \tilde{C}^{\rho\delta\theta|\xi\chi\tau;\rho'\delta'|\xi'\chi'}, \quad (151)$$

$$C^{\rho\delta\theta|\xi\chi\tau;\rho'\delta'|\xi'\chi'} = \tilde{C}^{\rho\delta\tau|\xi\chi\theta;\rho'\delta'|\xi'\chi'} + \tilde{C}^{\rho\delta\theta|\xi\chi\tau;\rho'\delta'|\xi'\chi'}, \quad (152)$$

$$\hat{C}^{\rho\delta\xi\tau|\chi\theta;\rho'\delta'|\xi'\chi'} = \bar{C}^{\rho\delta\xi\tau|\chi\theta;\rho'\delta'|\xi'\chi'} + \bar{C}^{\rho\delta\xi\theta|\chi\tau;\rho'\delta'|\xi'\chi'}, \quad (153)$$

$$\hat{C}^{\rho\delta\xi\theta|\chi\tau;\rho'\delta'|\xi'\chi'} = \bar{C}^{\rho\delta\xi\tau|\chi\theta;\rho'\delta'|\xi'\chi'} + \bar{C}^{\rho\delta\xi\theta|\chi\tau;\rho'\delta'|\xi'\chi'}. \quad (154)$$

Wherever two sets of indices are connected by a semicolon, it is understood that the corresponding tensor possesses independently the mixed symmetries with respect to the former and respectively the latter set. On the other hand, it is obvious that

$$\partial_\tau \partial_\theta E^{\rho\delta\tau|\xi\chi\theta} = \frac{1}{2} \partial_\tau \partial_\theta (E^{\rho\delta\tau|\xi\chi\theta} + E^{\rho\delta\theta|\xi\chi\tau}), \quad (155)$$

$$\partial_\tau \partial_\theta H^{\rho\delta\xi\tau|\chi\theta} = \frac{1}{2} \partial_\tau \partial_\theta (H^{\rho\delta\xi\tau|\chi\theta} + H^{\rho\delta\xi\theta|\chi\tau}), \quad (156)$$

so Eqs. (142)–(143) become

$$F_{\mu\nu\gamma|\alpha\beta\sigma} \frac{\partial \bar{U}^{\mu\nu\gamma|\alpha\beta\sigma}}{\partial r_{\rho\delta|\xi\chi}} + K_{\lambda\mu\nu\gamma|\alpha\sigma} \frac{\partial \bar{W}^{\lambda\mu\nu\gamma|\alpha\sigma}}{\partial r_{\rho\delta|\xi\chi}} = \\ = C^{\rho\delta\tau|\xi\chi\theta;\rho'\delta'|\xi'\chi'} \partial_\tau \partial_\theta r_{\rho'\delta'|\xi'\chi'} + C^{\rho\delta\theta|\xi\chi\tau;\rho'\delta'|\xi'\chi'} \partial_\tau \partial_\theta t_{\rho'\delta'|\xi'\chi'}, \quad (157)$$

$$F_{\mu\nu\gamma|\alpha\beta\sigma} \frac{\partial \bar{U}^{\mu\nu\gamma|\alpha\beta\sigma}}{\partial t_{\rho\delta|\xi\chi}} + K_{\lambda\mu\nu\gamma|\alpha\sigma} \frac{\partial \bar{W}^{\lambda\mu\nu\gamma|\alpha\sigma}}{\partial t_{\rho\delta|\xi\chi}} = \\ = \hat{C}^{\rho\delta\xi\tau|\chi\theta;\rho'\delta'|\xi'\chi'} \partial_\tau \partial_\theta r_{\rho'\delta'|\xi'\chi'} + \hat{C}^{\rho\delta\xi\tau|\chi\theta;\rho'\delta'|\xi'\chi'} \partial_\tau \partial_\theta t_{\rho'\delta'|\xi'\chi'}. \quad (158)$$

Taking the partial derivatives of Eqs. (157) and (158) with respect to $\partial_\tau \partial_\theta r_{\rho'\delta'|\xi'\chi'}$ and $\partial_\tau \partial_\theta t_{\rho'\delta'|\xi'\chi'}$, we infer the relations

$$\frac{\partial \bar{U}^{\mu\nu\gamma|\alpha\beta\sigma}}{\partial r_{\rho\delta|\xi\chi}} = k^{\mu\nu\gamma|\alpha\beta\sigma;\rho\delta|\xi\chi}, \quad \frac{\partial \bar{W}^{\lambda\mu\nu\gamma|\alpha\sigma}}{\partial r_{\rho\delta|\xi\chi}} = \bar{k}^{\lambda\mu\nu\gamma|\alpha\sigma;\rho\delta|\xi\chi}, \quad (159)$$

$$\frac{\partial \bar{U}^{\mu\nu\gamma|\alpha\beta\sigma}}{\partial t_{\rho\delta\xi|\chi}} = \hat{k}^{\mu\nu\gamma|\alpha\beta\sigma;\rho\delta\xi|\chi}, \quad \frac{\partial \bar{W}^{\lambda\mu\nu\gamma|\alpha\sigma}}{\partial t_{\rho\delta\xi|\chi}} = \tilde{k}^{\lambda\mu\nu\gamma|\alpha\sigma;\rho\delta\xi|\chi}, \quad (160)$$

where $k^{\mu\nu\gamma|\alpha\beta\sigma;\rho\delta\xi|\chi}$, $\bar{k}^{\lambda\mu\nu\gamma|\alpha\sigma;\rho\delta\xi|\chi}$, $\hat{k}^{\mu\nu\gamma|\alpha\beta\sigma;\rho\delta\xi|\chi}$, and $\tilde{k}^{\lambda\mu\nu\gamma|\alpha\sigma;\rho\delta\xi|\chi}$ denote some non-derivative, constant tensors. By means of relations (159) and (160) we obtain (up to some irrelevant constants)

$$\bar{U}^{\mu\nu\gamma|\alpha\beta\sigma} = k^{\mu\nu\gamma|\alpha\beta\sigma;\rho\delta\xi|\chi} r_{\rho\delta|\xi\chi} + \hat{k}^{\mu\nu\gamma|\alpha\beta\sigma;\rho\delta\xi|\chi} t_{\rho\delta\xi|\chi}, \quad (161)$$

$$\bar{W}^{\lambda\mu\nu\gamma|\alpha\sigma} = \bar{k}^{\lambda\mu\nu\gamma|\alpha\sigma;\rho\delta\xi|\chi} r_{\rho\delta|\xi\chi} + \tilde{k}^{\lambda\mu\nu\gamma|\alpha\sigma;\rho\delta\xi|\chi} t_{\rho\delta\xi|\chi}. \quad (162)$$

From the expression of ω_2 given by (137) we notice that the terms $k^{\mu\nu\gamma|\alpha\beta\sigma;\rho\delta\xi|\chi} r_{\rho\delta|\xi\chi}$ and $\tilde{k}^{\lambda\mu\nu\gamma|\alpha\sigma;\rho\delta\xi|\chi} t_{\rho\delta\xi|\chi}$ appearing in (161) and (162) bring no contributions to cross-interactions. For this reason, we take

$$k^{\mu\nu\gamma|\alpha\beta\sigma;\rho\delta\xi|\chi} = 0, \quad \tilde{k}^{\lambda\mu\nu\gamma|\alpha\sigma;\rho\delta\xi|\chi} = 0, \quad (163)$$

such that (up to a total, irrelevant divergence) ω_2 takes the form

$$\omega_2 = \bar{k}^{\lambda\mu\nu\gamma|\alpha\sigma;\rho\delta\xi|\chi} K_{\lambda\mu\nu\gamma|\alpha\sigma} r_{\rho\delta|\xi\chi} + \hat{k}^{\mu\nu\gamma|\alpha\beta\sigma;\rho\delta\xi|\chi} F_{\mu\nu\gamma|\alpha\beta\sigma} t_{\rho\delta\xi|\chi}. \quad (164)$$

The most general expression of $\bar{k}^{\lambda\mu\nu\gamma|\alpha\sigma;\rho\delta\xi|\chi}$ is represented by

$$\begin{aligned} \bar{k}^{\lambda\mu\nu\gamma|\alpha\sigma;\rho\delta\xi|\chi} = & \kappa \left[\frac{1}{4} \varepsilon^{\lambda\mu\nu\gamma\rho\delta} (\sigma^{\xi\alpha} \sigma^{\chi\sigma} - \sigma^{\xi\sigma} \sigma^{\chi\alpha}) \right. \\ & + \frac{1}{4} \varepsilon^{\lambda\mu\nu\gamma\xi\chi} (\sigma^{\rho\alpha} \sigma^{\delta\sigma} - \sigma^{\rho\sigma} \sigma^{\delta\alpha}) \\ & \left. - \frac{1}{24} \varepsilon^{\lambda\mu\nu\gamma[\rho\delta} \delta_{\tau}^{\xi} \delta_{\theta}^{\chi]} (\sigma^{\tau\alpha} \sigma^{\theta\sigma} - \sigma^{\tau\sigma} \sigma^{\theta\alpha}) \right], \quad (165) \end{aligned}$$

which then yields

$$\bar{k}^{\lambda\mu\nu\gamma|\alpha\sigma;\rho\delta\xi|\chi} K_{\lambda\mu\nu\gamma|\alpha\sigma} r_{\rho\delta|\xi\chi} = \kappa \varepsilon^{\lambda\mu\nu\gamma\rho\delta} r_{\rho\delta|\xi\chi} K_{\lambda\mu\nu\gamma|\alpha\sigma}^{\xi\chi}, \quad (166)$$

with κ a real constant. On the other hand, there exist non-trivial constant tensors of the type $\hat{k}^{\mu\nu\gamma|\alpha\beta\sigma;\rho\delta\xi|\chi}$, but they all lead in the end to

$$\hat{k}^{\mu\nu\gamma|\alpha\beta\sigma;\rho\delta\xi|\chi} F_{\mu\nu\gamma|\alpha\beta\sigma} t_{\rho\delta\xi|\chi} \equiv 0 \quad (167)$$

due to the algebraic Bianchi I identity $F_{[\mu\nu\gamma|\alpha]\beta\sigma} = 0$. Such constants have an intricate and non-illuminating form, and therefore we will skip them. Inserting (166) and (167) in (164), we deduce

$$\omega_2 = \kappa \varepsilon^{\lambda\mu\nu\gamma\rho\delta} r_{\rho\delta|\xi\chi} K_{\lambda\mu\nu\gamma|\alpha\sigma}^{\xi\chi}. \quad (168)$$

Acting with γ on (168), it is easy to see that

$$\gamma\omega_2 = -2\kappa\varepsilon^{\lambda\mu\nu\gamma\rho\delta} \left(\partial_{[\lambda} K_{\mu\nu\gamma]}^{\xi} \right) \mathcal{C}_{\rho\delta|\xi} \neq \partial_\mu j_2^\mu, \quad (169)$$

where $K_{\mu\nu\gamma|\tau}$ is the trace of the curvature tensor $K_{\mu\nu\gamma\alpha|\tau\beta}$, $K_{\mu\nu\gamma|\tau} = \sigma^{\alpha\beta} K_{\mu\nu\gamma\alpha|\tau\beta}$. It is worthy to notice that $\gamma\omega_2 \neq \partial_\mu j_2^\mu$ follows from the differential Bianchi II identity $\partial_\beta K_{\lambda\mu\nu\gamma}^{\beta\xi} = \partial_{[\lambda} K_{\mu\nu\gamma]}^{\xi}$. Due to (169), we must take

$$\kappa = 0, \quad (170)$$

and hence

$$\omega_2 = 0. \quad (171)$$

Replacing (120), (127), and (171) in (115), we finally have that

$$\bar{a}_0^{\text{t-r}} = 0 \quad (172)$$

in (113).

Putting together the results expressed by formulae (77), (86), (96), (110)–(111), (113), and (172), we can state that the most general form of the first-order deformation associated with the free theory (1) reads as

$$\begin{aligned} S_1 = & \int d^6x \left[\varepsilon_{\mu\nu\alpha\lambda\beta\gamma} \eta^{*\mu\nu\alpha} \partial^\lambda \mathcal{C}^{\beta\gamma} \right. \\ & - 2\varepsilon_{\lambda\mu\nu\rho\beta\gamma} t^{*\lambda\mu\nu|\alpha} \left(\partial^\rho \mathcal{C}^{\beta\gamma|}_{\alpha} - \frac{1}{4} \delta^\gamma_\alpha \partial^{[\rho} \mathcal{C}^{\beta\tau]}_{\tau} \right) \\ & \left. - 2\varepsilon^{\lambda\mu\nu\alpha\beta\gamma} t_{\lambda\mu\nu|\rho} \left(\partial_\sigma \partial_\alpha r_{\beta\gamma}^{\sigma\rho} - \frac{1}{2} \delta^\rho_\gamma \partial^\tau \partial_\alpha r_{\beta\tau} \right) + r \right]. \quad (173) \end{aligned}$$

5.2 Higher-order deformations

In the sequel we approach the higher-order deformation equations. The second-order deformation is controlled by Eq. (47). After some computations we arrive at

$$(S_1, S_1) = s \int d^6x \left(10r^{\lambda\rho|[\alpha\beta,\gamma]} r_{\lambda\rho|[\alpha\beta,\gamma]} - 12r_{\lambda\rho|}^{[\alpha\beta,\rho]} r^{\lambda\sigma|}_{[\alpha\beta,\sigma]} \right), \quad (174)$$

such that

$$S_2 = \int d^6x \left(-5r^{\lambda\rho|[\alpha\beta,\gamma]} r_{\lambda\rho|[\alpha\beta,\gamma]} + 6r_{\lambda\rho|}^{[\alpha\beta,\rho]} r^{\lambda\sigma|}_{[\alpha\beta,\sigma]} \right). \quad (175)$$

Using (173)–(175) in (48), we determine the third-order deformation as

$$S_3 = 0. \quad (176)$$

Under these conditions, it is easy to see that all the remaining higher-order deformation equations are fulfilled with the choice

$$S_k = 0, \quad k > 3. \quad (177)$$

Inserting now relations (42) (for $D = 6$), (173), (175), (176), and (177) into (43), we find nothing but formula (50), which then implies relations (52)–(57). This ends the proof of the main result.

6 Conclusion

In this paper we have developed a cohomological approach to the problem of constructing consistent interactions between a massless tensor gauge field with the mixed symmetry $(3, 1)$ and a purely spin-two field with the mixed symmetry $(2, 2)$. Under the general assumptions of analyticity of the deformations in the coupling constant, locality, (background) Lorentz invariance, Poincaré invariance, and the requirement that the interaction vertices contain at most two spatiotemporal derivatives of the fields, we have exhausted all the consistent, non-trivial couplings. Our final result is rather surprising since it enables non-trivial cross-couplings between the dual formulation of linearized gravity in six spatiotemporal dimensions and the tensor field with the mixed symmetry of the Riemann tensor. Although the cross-couplings break the PT invariance and are merely mixing-component terms, still this is the first situation encountered so far where the gauge transformations and reducibility functions in the $(3, 1)$ sector are modified with respect to the free ones.

Acknowledgments

One of the authors (E.M.B.) acknowledges financial support from the contract 464/2009 in the framework of the programme IDEI of C.N.C.S.I.S. (Romanian National Council for Academic Scientific Research).

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